2009 NCN@Purdue-Intel Summer School
Notes on Percolation and Reliability Theory

Lecture 2
Thresholds, Islands, and Fractals

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outline of lecture 2

1) **Basic concepts of percolation theory**
2) Percolation threshold and ‘excluded volume’
3) Cluster size distribution, cluster Radius
4) Fractal dimension of a random surface
5) Conclusion

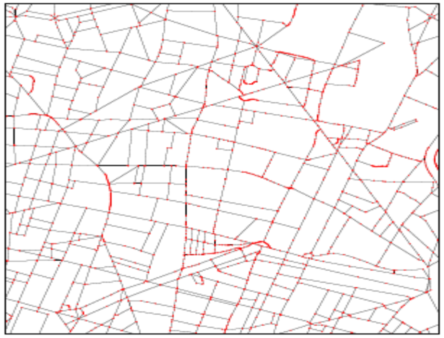
Application Notes: Nanocrystal Flash

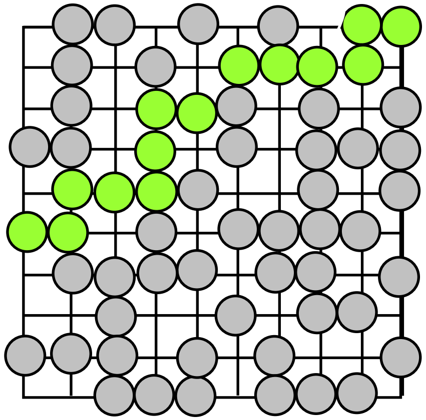
three concepts of random systems

Percolation threshold

epidemics, forest fire, telecom grid, www

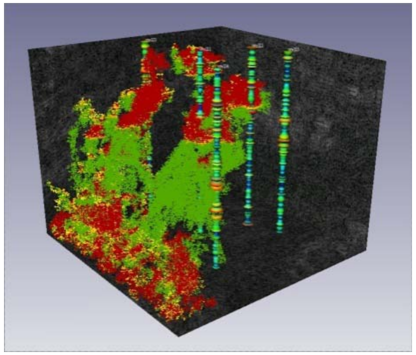
Nanonets, photovoltaics

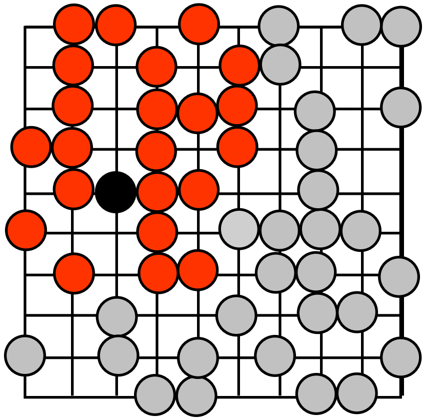




Cluster sizes

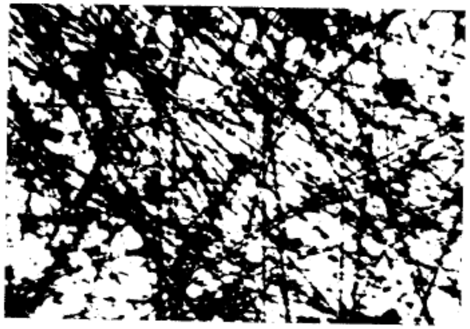
Oil fields, NC Flash

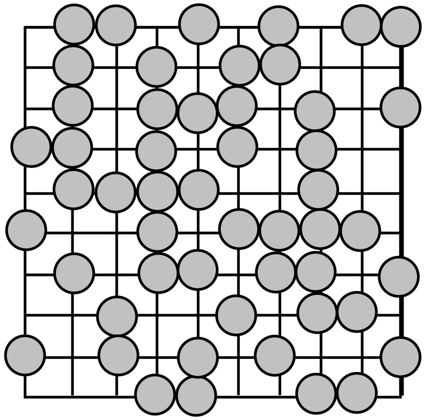




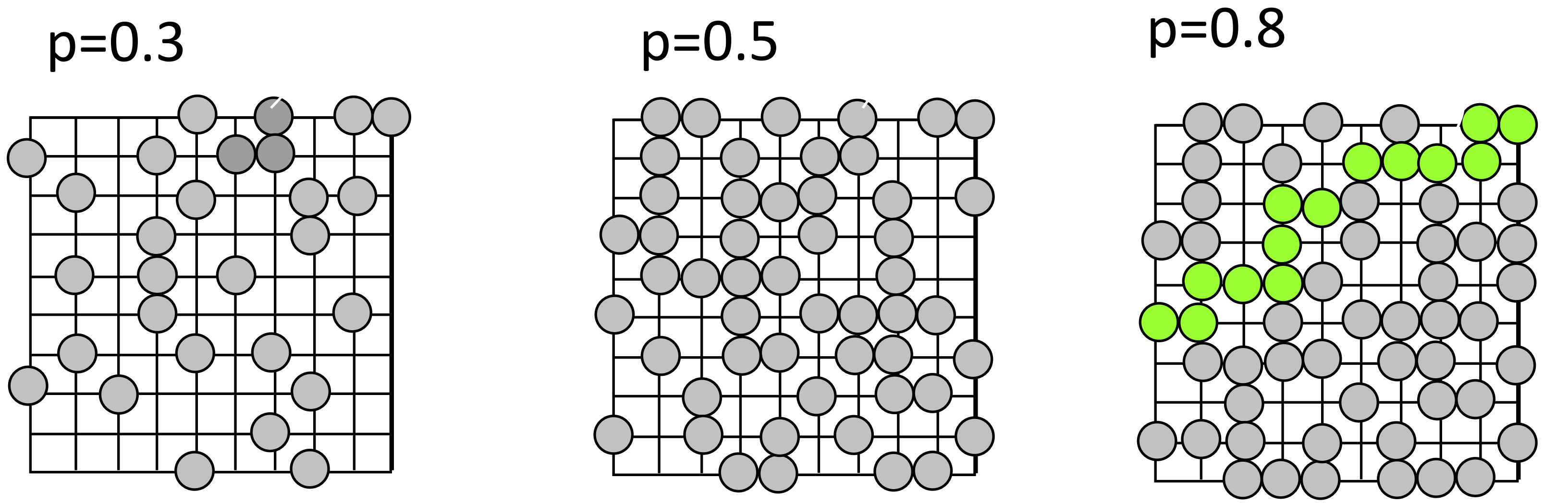
Fractal dimension

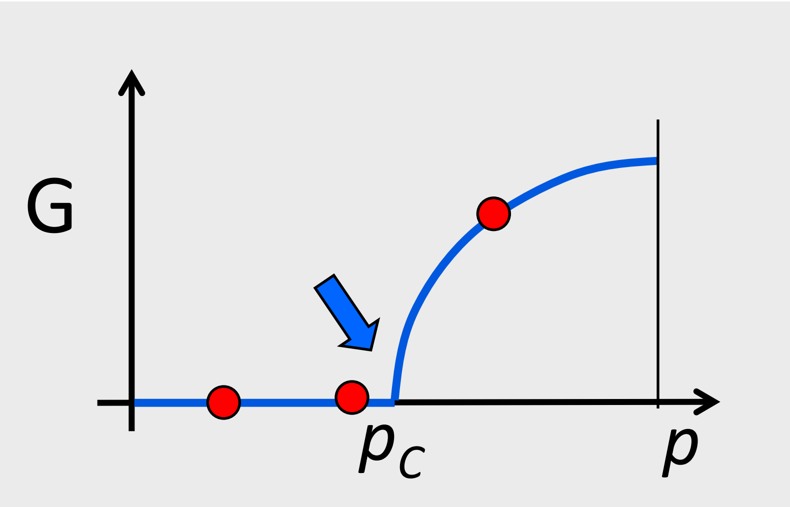
Aerosol, paper, sensors





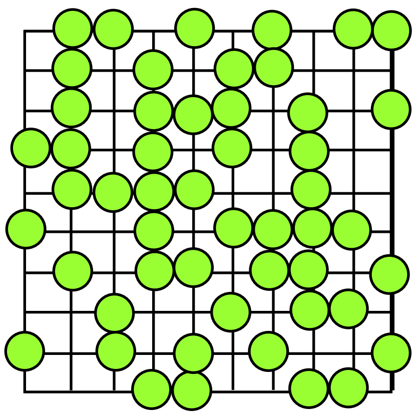
basic concepts: percolation threshold





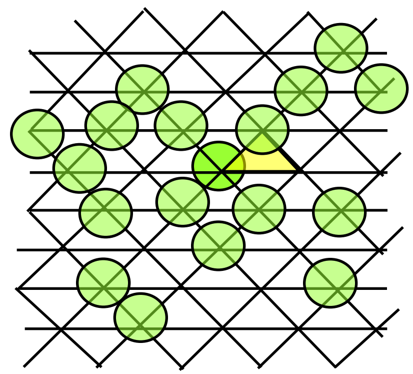
calculation of percolation threshold

Square



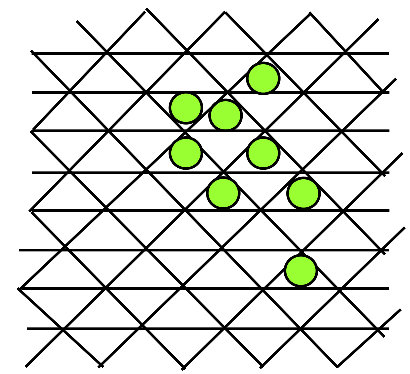
$p_c = 0.593$

Triangular



$p_c = 0.500$

Hexagonal

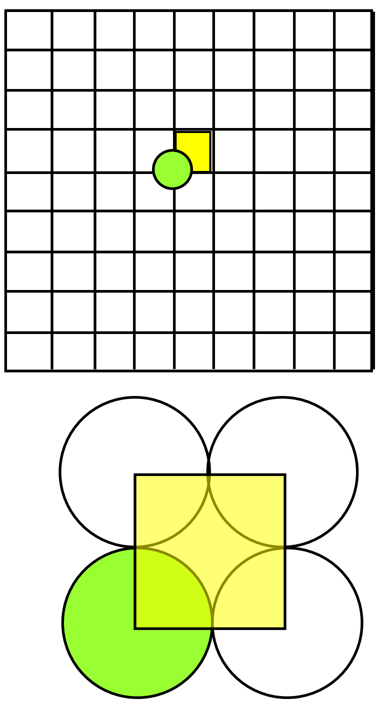


$p_c = 0.697$

Percolation threshold ($p_c = N_c / N_T$) depends on lattice,
there is something wrong here !

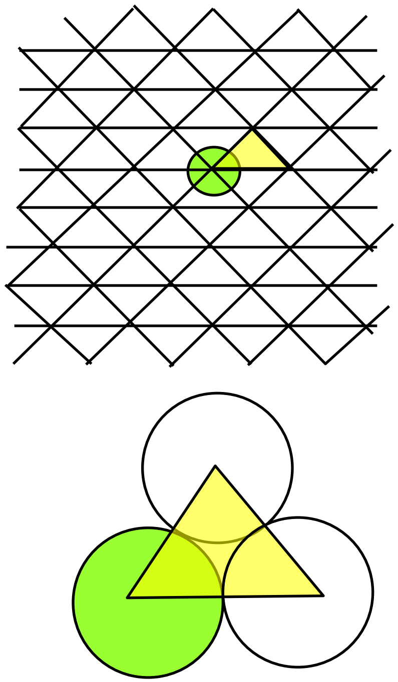
area fraction fill-factor $F \equiv A_{element} / A_{cell}$

Square



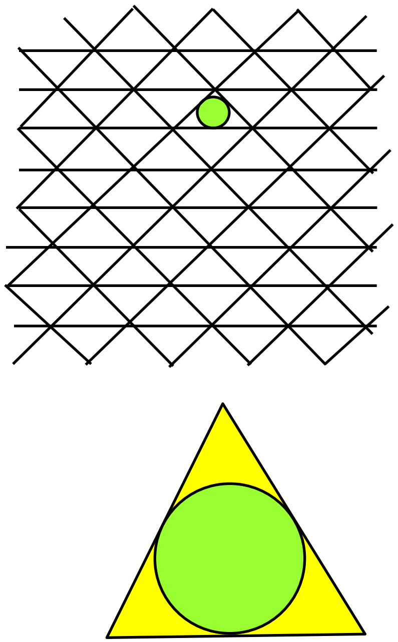
$$F = \frac{\pi}{4}$$

Triangular



$$F = \frac{\pi}{2\sqrt{3}}$$

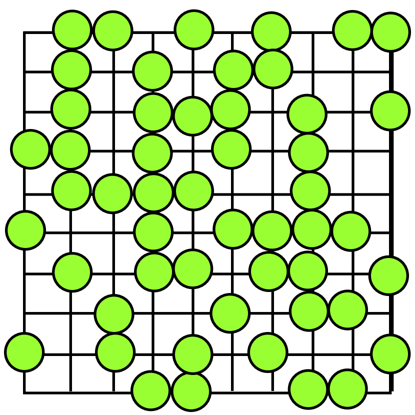
Hexagonal



$$F = \frac{\pi}{3\sqrt{3}}$$

$(F \times p_c)$ is universal ....

Square

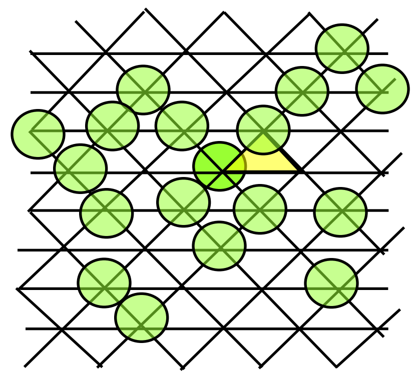


$p_c = 0.593$

$$F = \frac{\pi}{4}$$

$Fp_c \sim 0.45$

Triangular

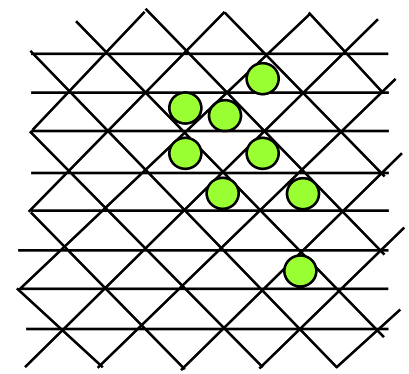


$p_c = 0.500$

$$F = \frac{\pi}{2\sqrt{3}}$$

$Fp_c \sim 0.45$

Hexagonal



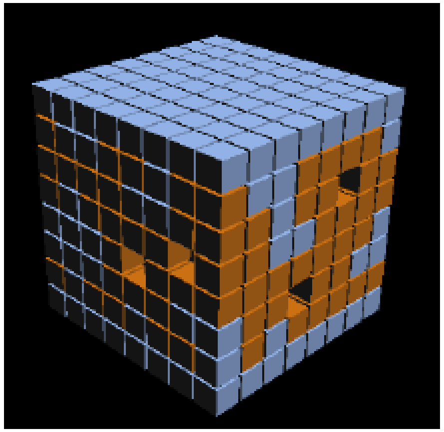
$p_c = 0.697$

$$F = \frac{\pi}{3\sqrt{3}}$$

$Fp_c \sim 0.42$

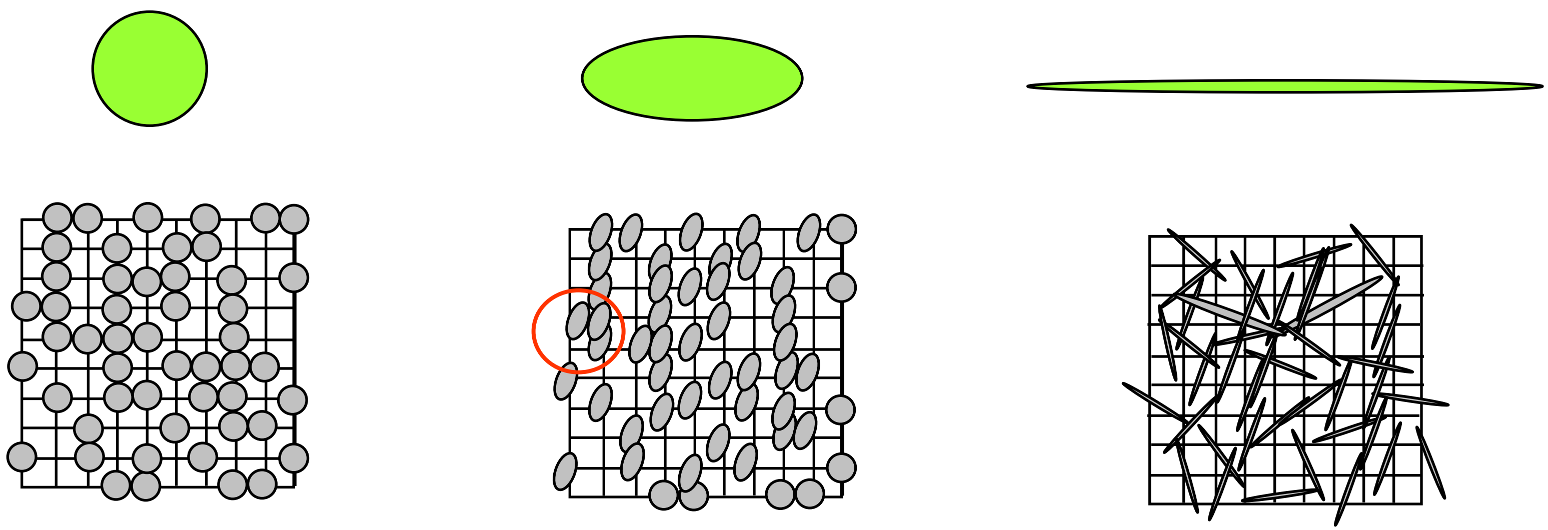
hw: Percolation in 3D lattices

For simple cubic lattice site percolation, $Fp_c \sim 0.16$ and $p_c \sim 0.311$. Here F is the volume fill fraction, not area fill fraction.



Use the universality of Fp_c to show that the percolation threshold for FCC lattice must be approximately 0.1

percolation involving other shapes

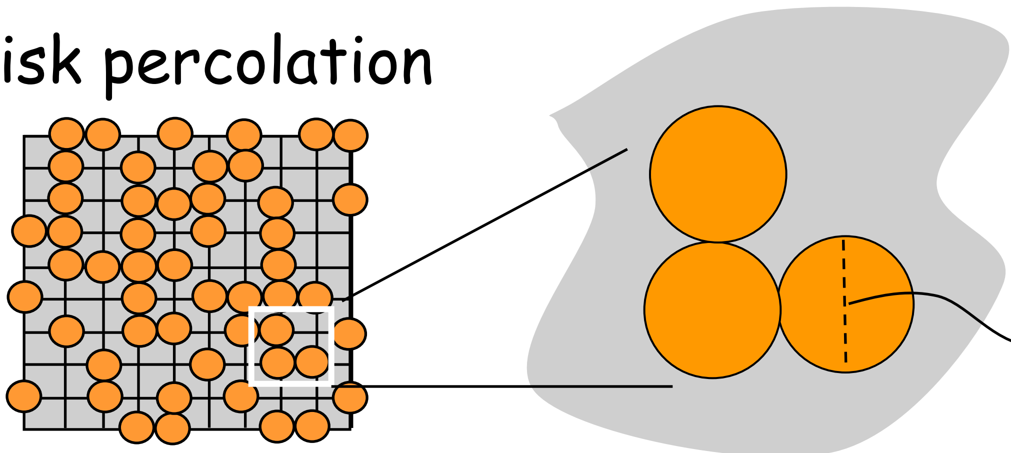


How do I determine the percolation threshold?

F p_c ~ 0.45 will not work, unfortunately, because sticks have zero area, i.e. F~0 !

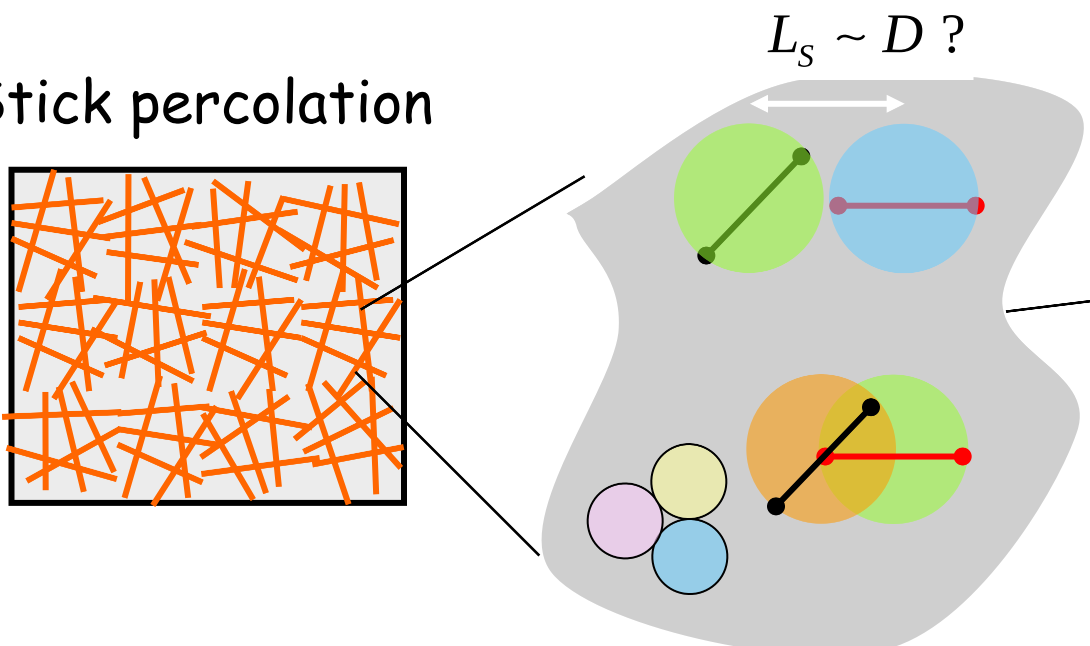
excluded area … first an intuitive result

Disk percolation



$$N_C \approx \frac{1}{\frac{\pi D^2}{4}} = \frac{4}{\pi D^2}$$

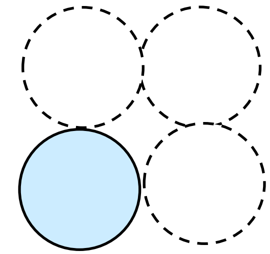
Stick percolation



$$N_C \approx \frac{1}{\frac{\pi \left(L_S/2\right)^2}{4}} = \frac{4^2}{\pi L_S^2}$$

$$N_{C,exact} = \frac{4.26^2}{\pi L_S^2}$$

the concept of excluded area .....



For disks on arbitrary grid …

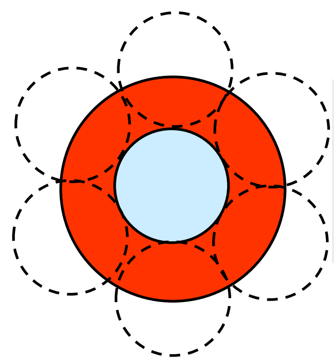
$$F \times p_c = \frac{A_{element}}{A_{cell}} \bullet \frac{N_C}{N_T} \approx 0.45$$

$$A_{element} = \frac{\pi}{4}(D)^2$$

X 4

For arbitrary shape on arbitrary grid …

$$\frac{A_{ex}}{A_{cell}} \bullet \frac{N_C}{N_T} \approx$$

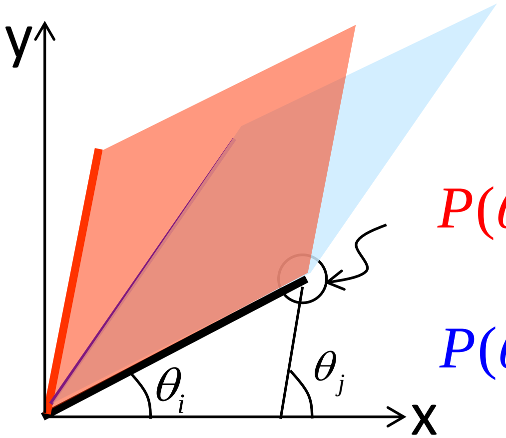


$$A_{ex} = \frac{\pi}{4}(2 \bullet D)^2$$

Percolation begins when excluded volume is routinely breached

excluded volume for a stick ...

$$A_{\theta_i,\theta_j} = L_S \; L_S \; \sin(\theta_i - \theta_j)$$



$$P(\theta_i) = \frac{1}{\pi}$$

$$P(\theta_j) = \frac{1}{\pi}$$

$$A_{ex} = \int_{-\pi/2}^{\pi/2} d\theta_i P(\theta_i) \int_{-\pi/2}^{\pi/2} d\theta_j P(\theta_j) \times A_{\theta_i,\theta_j}$$

$$= \frac{2}{\pi} L_S^2$$

excluded area for a stick ....

$$\frac{A_{ex}}{A_{cell}} \bullet \frac{N_C}{N_T} \approx 1.8$$

$$\left(\frac{2L_S^2}{\pi} \middle/ A_{cell} \right) \bullet \frac{N_C}{N_T} \approx 1.8$$

$$N_C \approx \frac{0.9\pi}{L_S^2} \simeq \frac{3^2}{\pi L_S^2} \text{ area}^{-1}$$

$$N_{C,exact} \approx \frac{4.26^2}{\pi L_S{}^2}$$

percolation density correct within a factor of 2 !

hw: excluded volume for other shapes ..

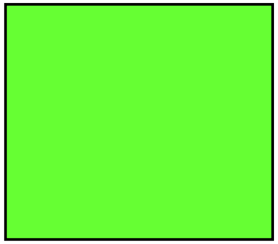
curved stick ...

Ans. $$A_{ex} = \frac{2}{\pi} L_{chord}^2$$

$$(\text{if } L_{chord} < \frac{R}{2})$$

Hint.
Use the stick algorithm

square ...

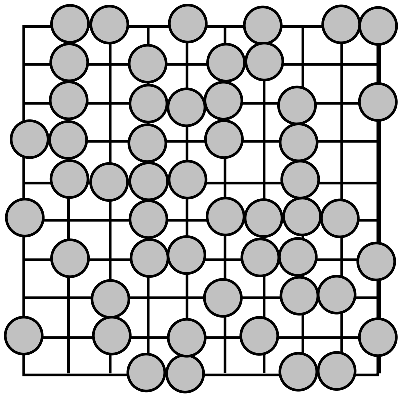


$$A_{ex} = 2L^2 \left(1 + 2/\pi + 4/\pi^2\right)$$

Hint. Compare with circle

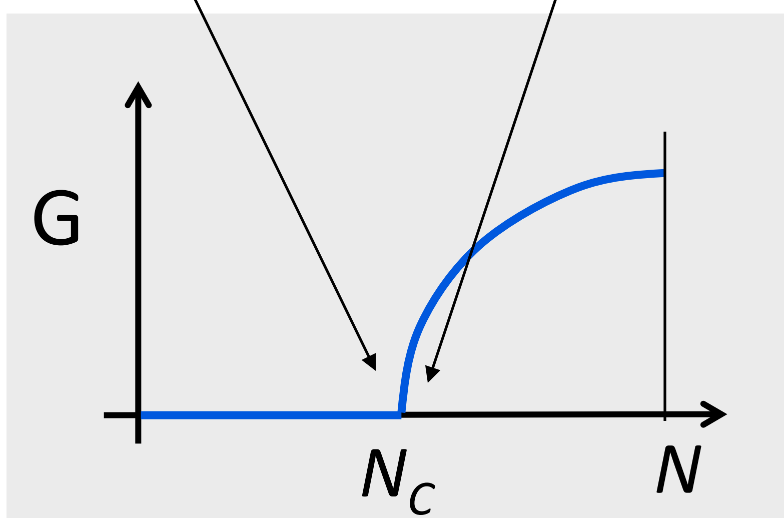
For general shape, use the Monte Carlo code posted

summary: percolation threshold



$$N_C \approx \frac{4}{\pi D^2}$$

$$N_C \approx \frac{4.26^2}{\pi {L_S}^2}$$



$$p \equiv \frac{N}{N_T}$$

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three topics of random systems

Percolation threshold

epidemics, forest fire,
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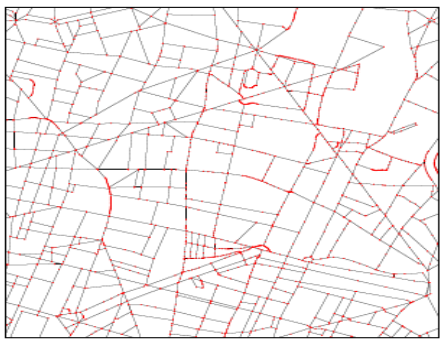
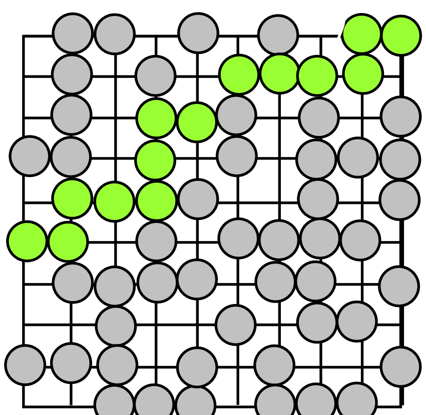
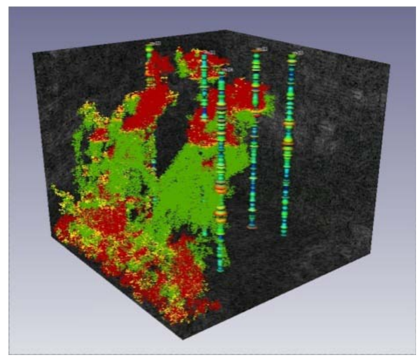
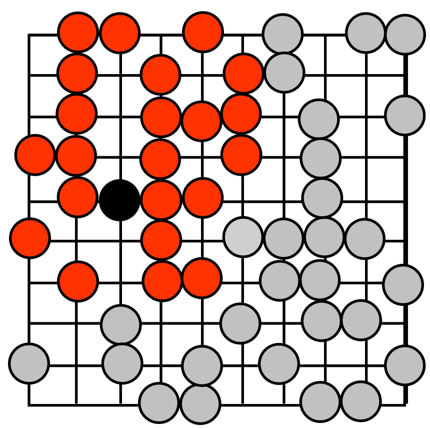




Cluster sizes

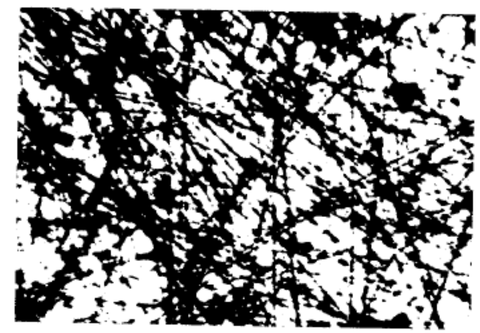
Oil fields, NC Flash

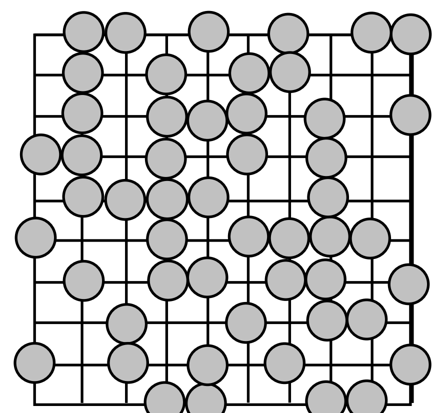




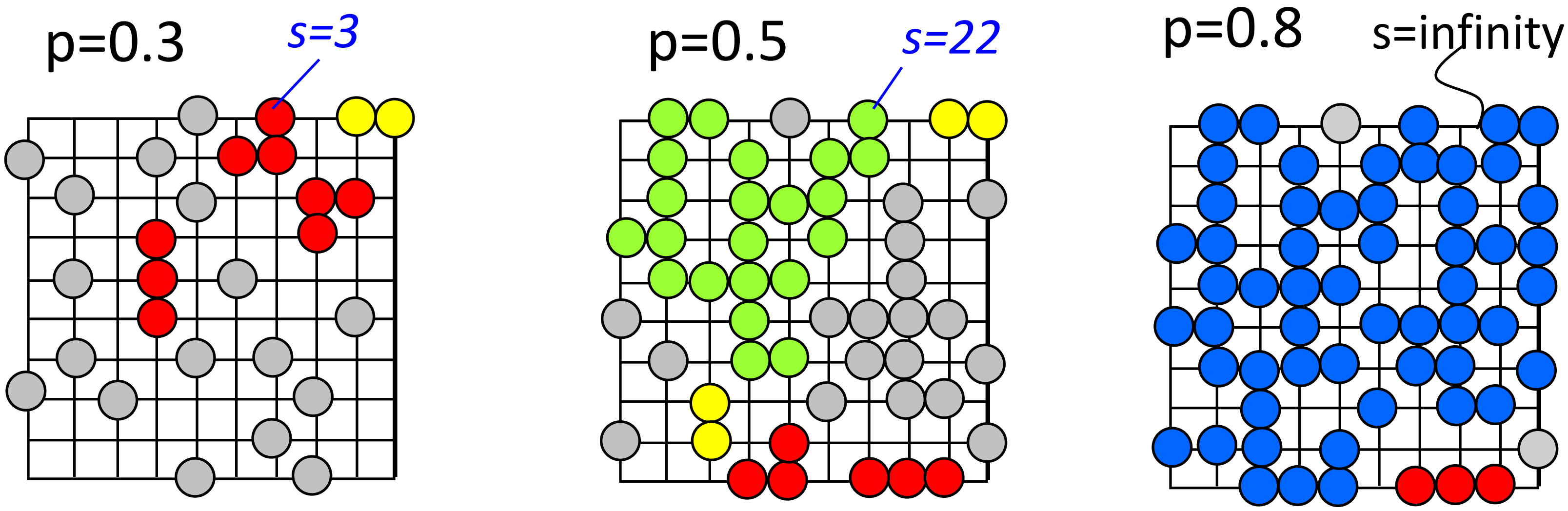
Fractal dimension

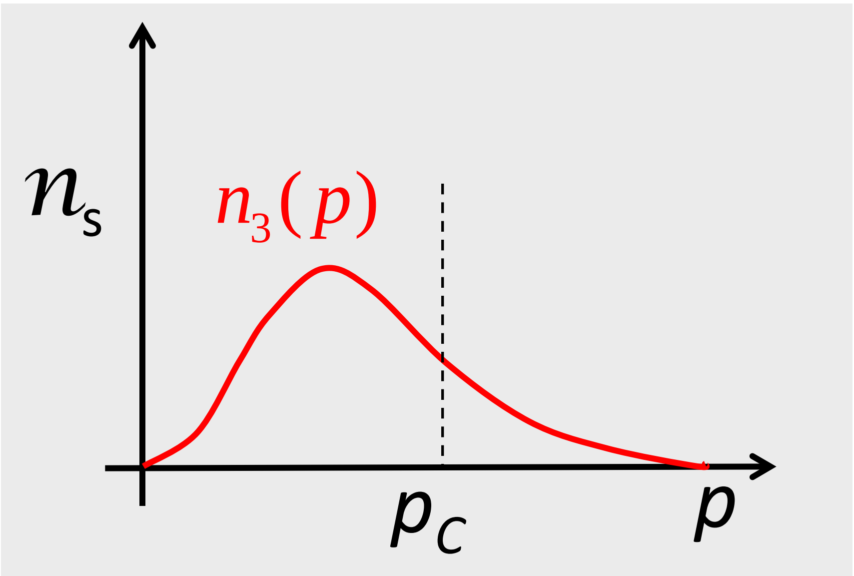
Aerosol, paper, sensors





basic concepts: cluster size

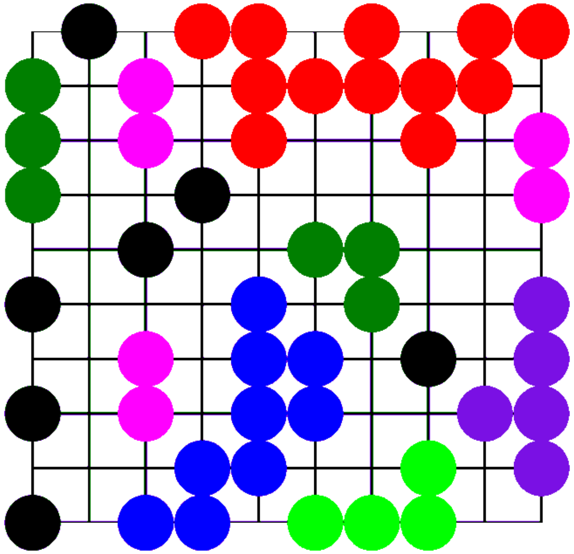




cluster-size distribution and its moments

$n_s(p)$

Number of cluster of size s divided by the number of sites



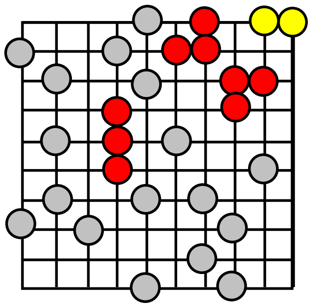
$$p = \sum_{0<s<\infty} s \times n_s(p)$$

$$s_{avg} = \frac{\sum_{s>0} s^2 n_s(p)}{\sum_{s>0} s n_s(p)}$$

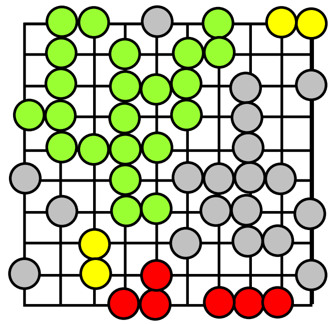
...plays a role similar to Boltzmann distribution $f(E, E_F)$

average cluster vs. infinite cluster

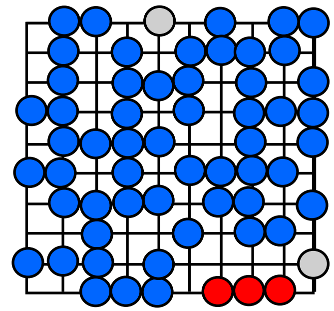
p=0.3



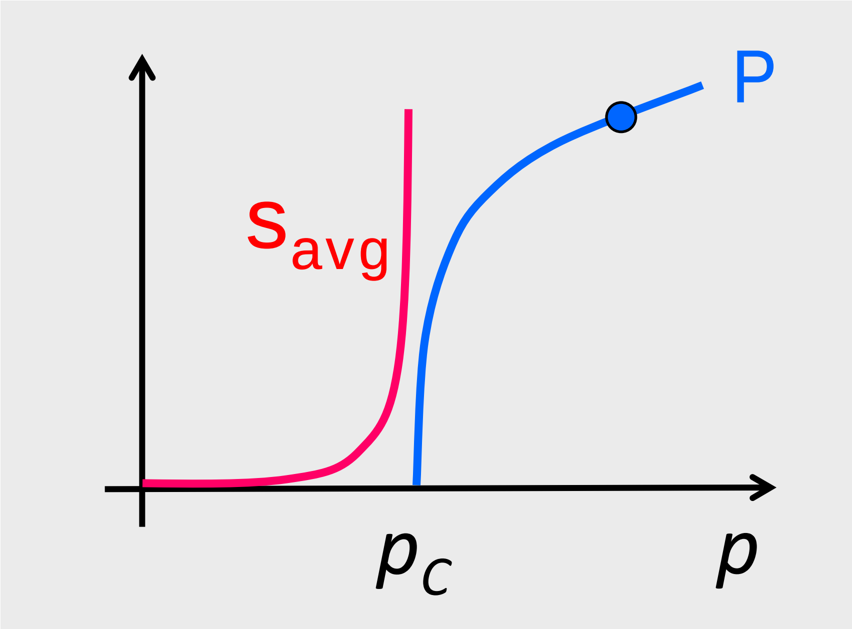
p=0.59



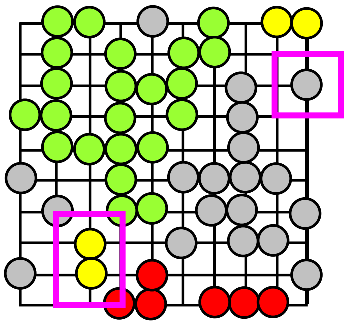
p=0.8

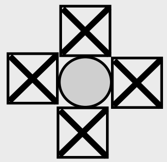


$$s_{avg} = \frac{\sum_{s>0} s^2 n_s(p)}{\sum_{s>0} s n_s(p)}$$

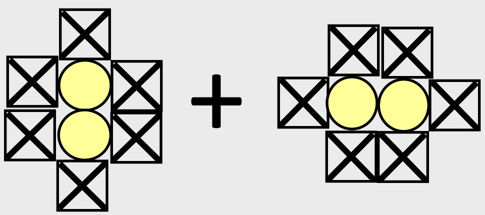


small-cluster size distribution

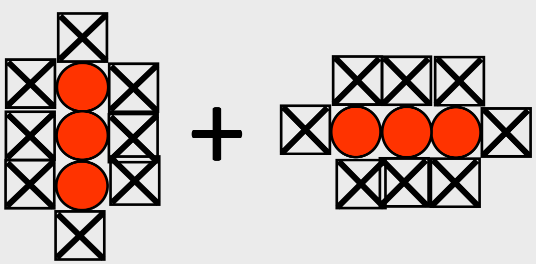




$$n_1(p) = 1 \times p \times (1-p)^4$$

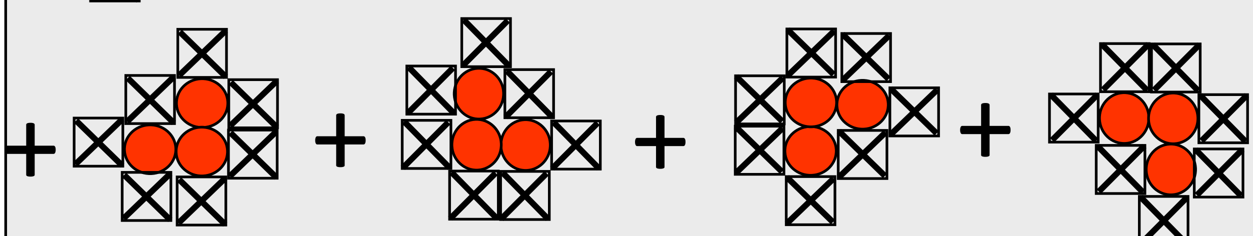


$$n_2(p) = 2 \times p^2 \times (1-p)^6$$



$$n_3(p) = 2 \times p^3 \times (1-p)^8$$

$$+\, 4 \times p^3 \times (1-p)^7$$



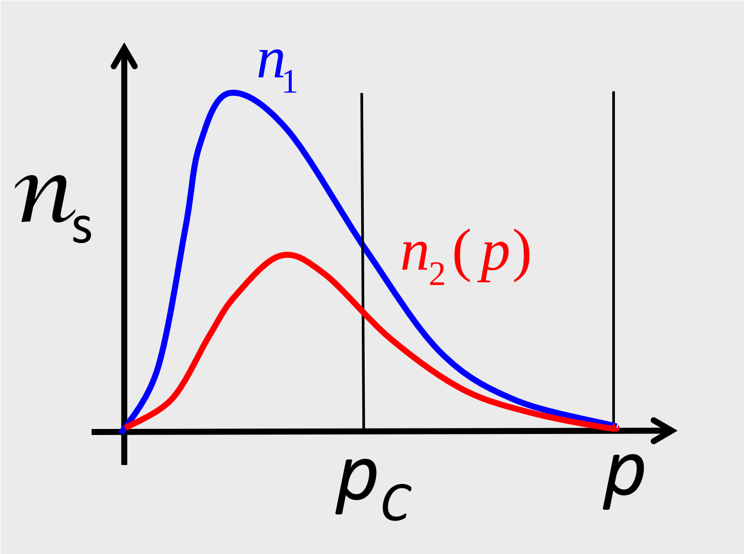
features of cluster-size distribution

$$n_1 = 1 \times p \times (1-p)^4$$

$$n_2 = 2 \times p^2 \times (1-p)^6$$

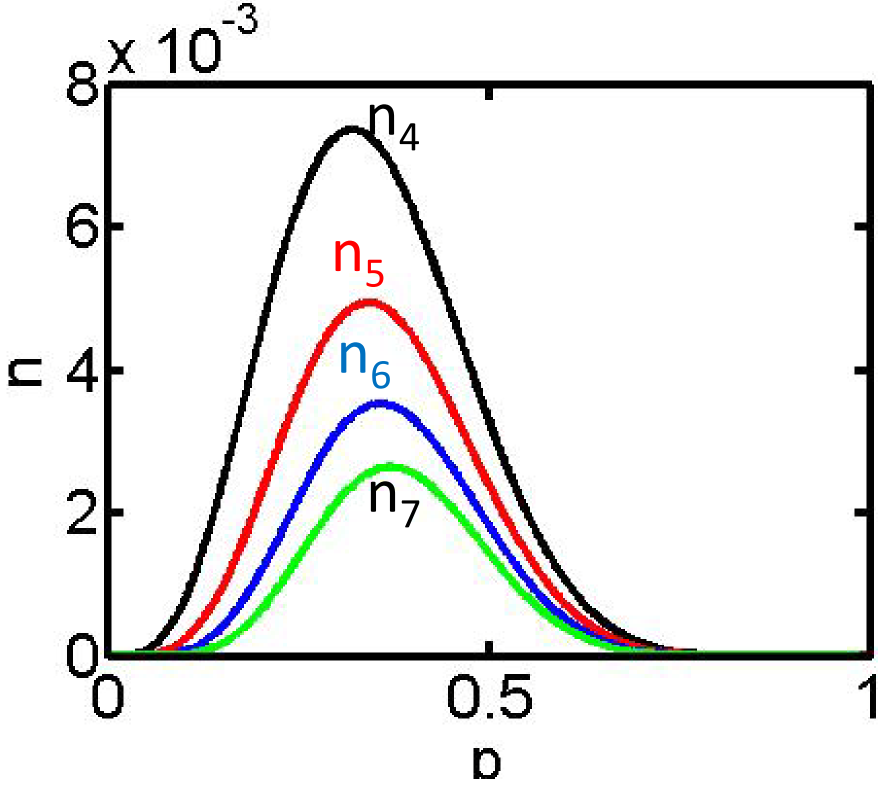
$$n_3 = 2 \times p^3 \times (1-p)^8$$
$$+4 \times p^3 \times (1-p)^7$$

- 'Zeros' at p=0,1 with single peak
- Peak shifts towards p_c with s (e.g. s=1 is 0.2, s=2 is 0.25; s=3 is 0.29)
- General form: $n_s(p) = \sum_t g_{st} \times p^s \times (1-p)^t$: g_{st} increases exponentially.

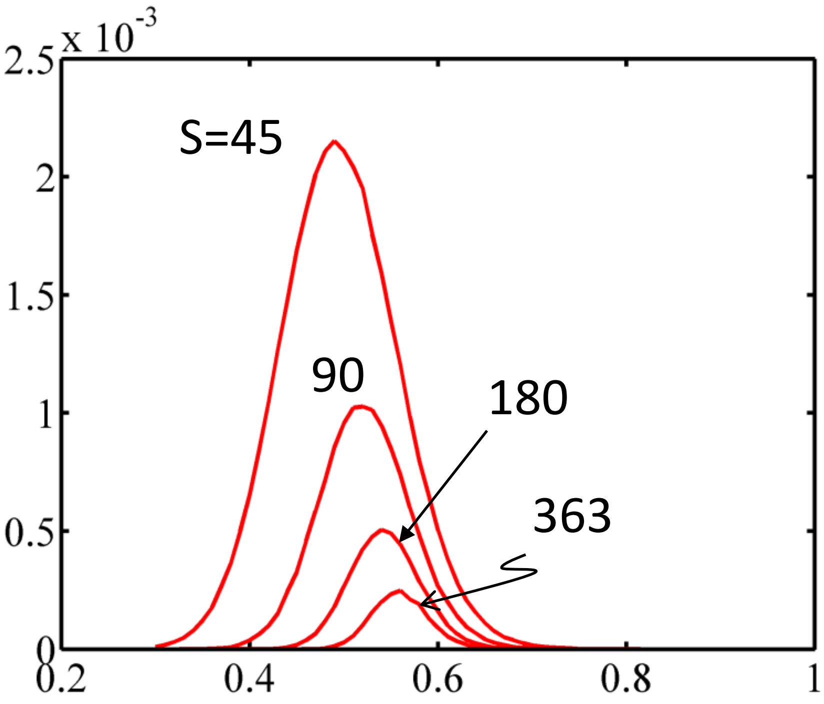


numerical plots for cluster-size distribution

Analytically …

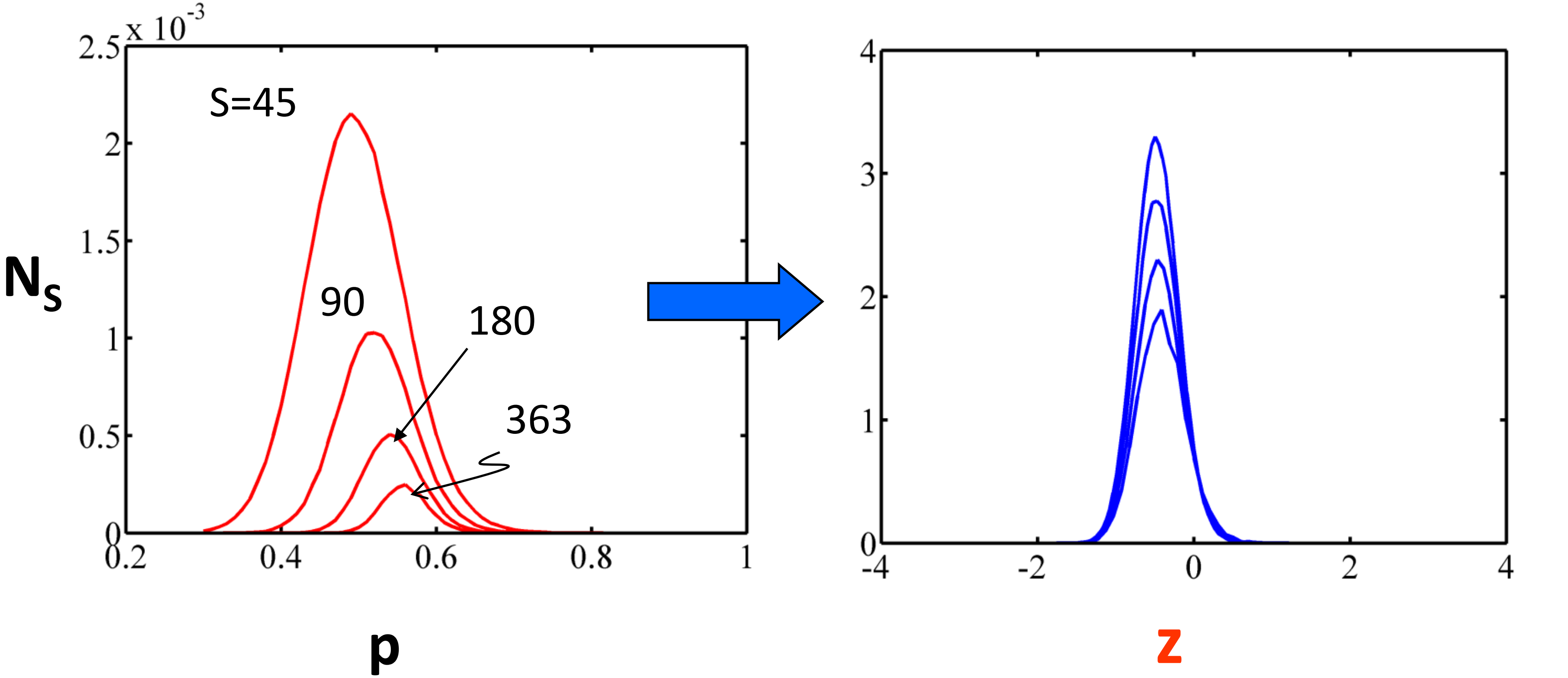


by computer …



$$\left[p_{\max}(s) - p_c\right] \times s^{0.395} \approx 0.45 \quad p_c \sim 0.6$$

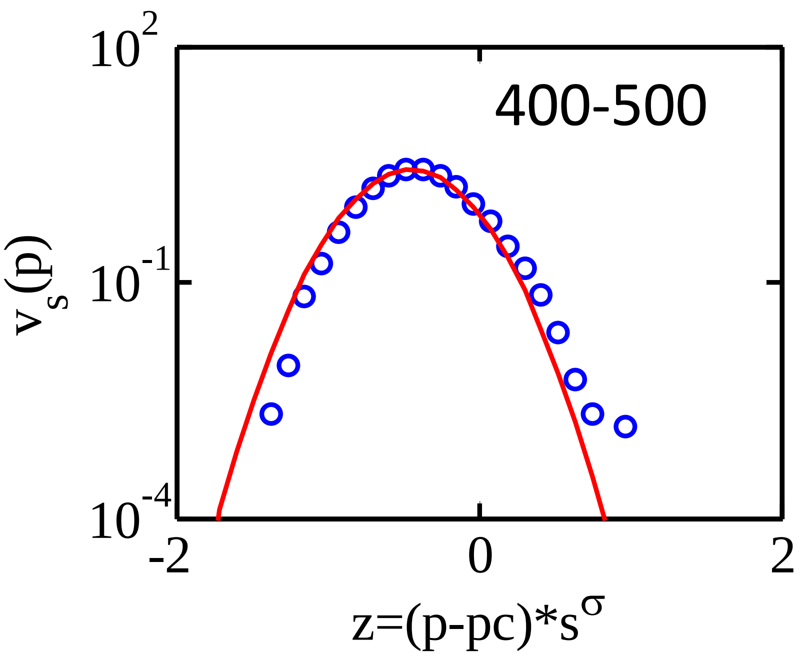
scaling of cluster sizes

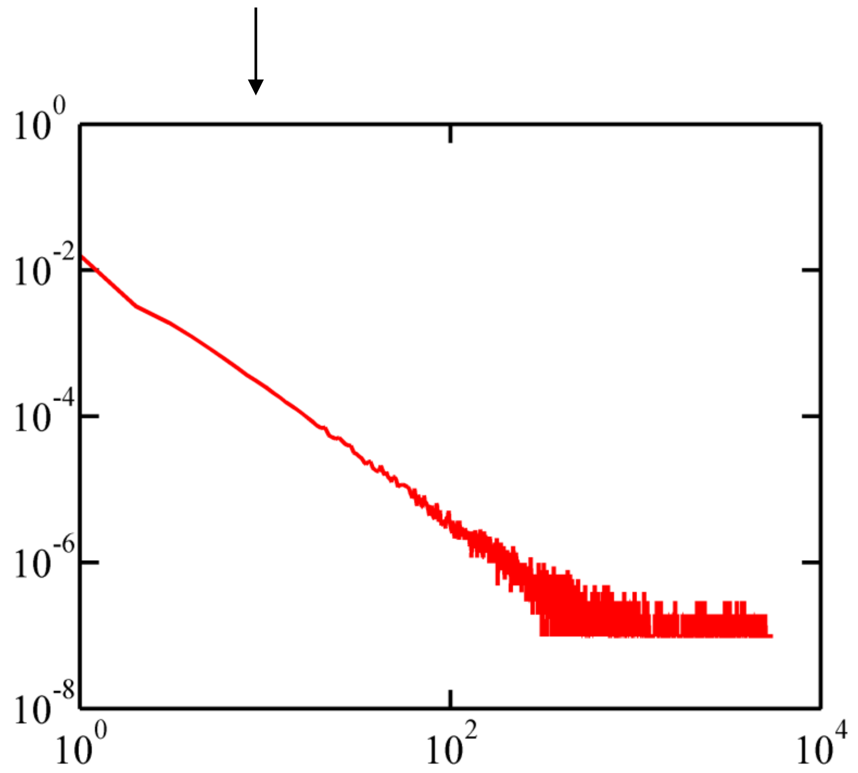


$$z \equiv \left[p(s) - p_c \right] s^{0.395}$$

for reasonably large cluster-sizes (s>20)

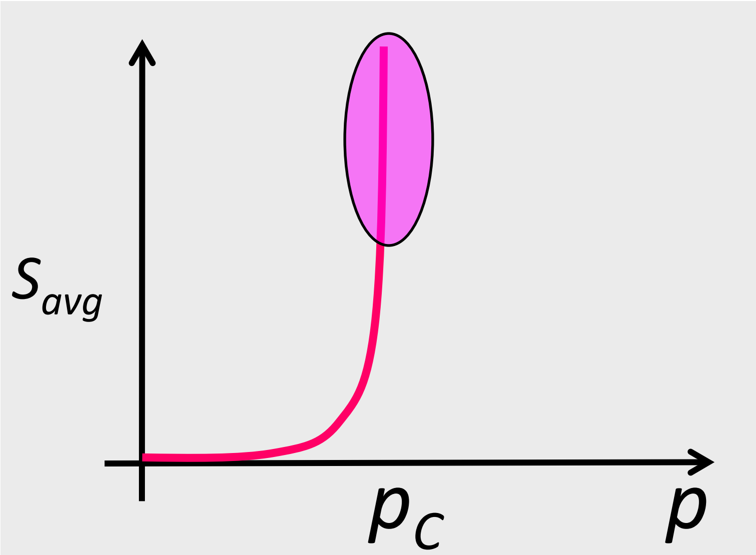
$$n_s(p) \approx Ae^{-c\left[(p-p_c)s^{\sigma}+0.45\right]^{\alpha}} \qquad n_s(p_c)$$





average cluster-size distribution

$$n_s(p) \approx Ae^{-c\left[(p-p_c)s^{\sigma}+0.45\right]^{\alpha}} \quad n_s(p_c) \leftarrow A/s^{-(2+\varepsilon)}$$

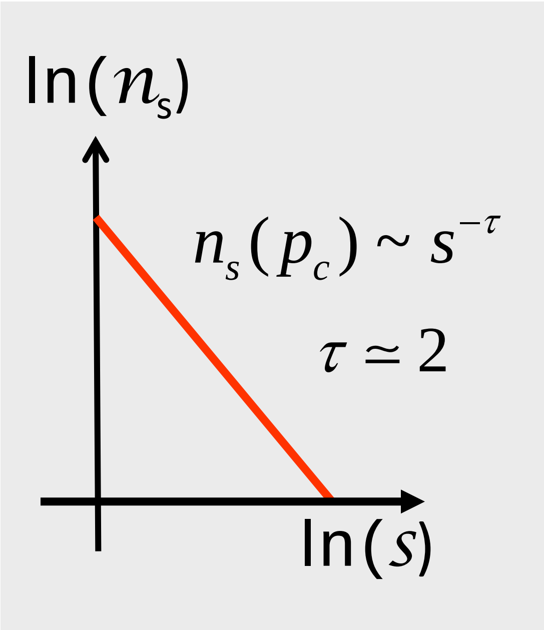


$$s_{avg} = \frac{\sum_{s>0} s^2 n_s(p)}{\sum_{s>0} s n_s(p)}$$

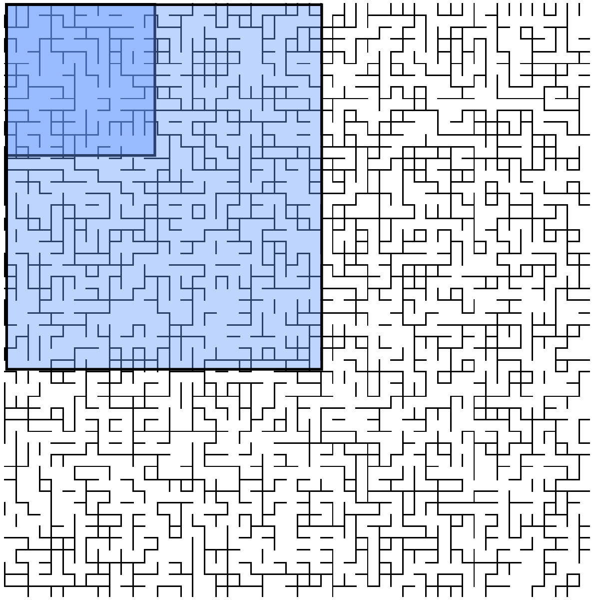
$$s_{avg}(p_c) = \frac{\int_1^{\infty} s^2 n_s(p_c) ds}{p} \rightarrow \infty$$

self-similarity and scale-invariance

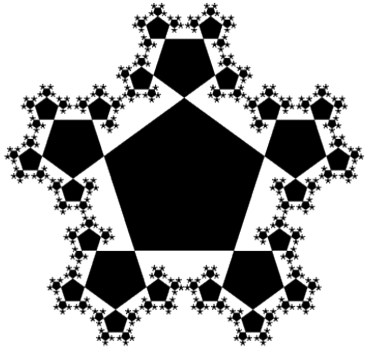
self-similarity

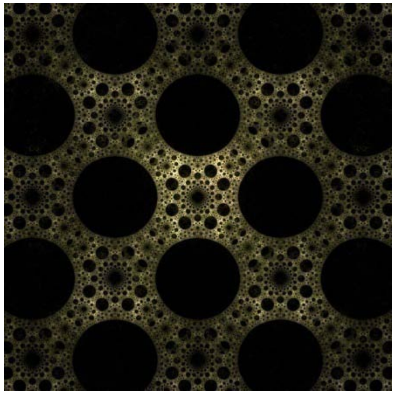


irregular
self-similarity



regular
self-similarity





This is the origin of the power of the percolation theory …

outline of lecture 2

1) Basic concepts of percolation theory
2) Percolation threshold and 'excluded volume'
3) Cluster size distribution, cluster Radius
4) Fractal dimension of a random surface
5) Conclusion

Application Notes: Nanocrystal Flash

three concepts of random systems

Percolation threshold

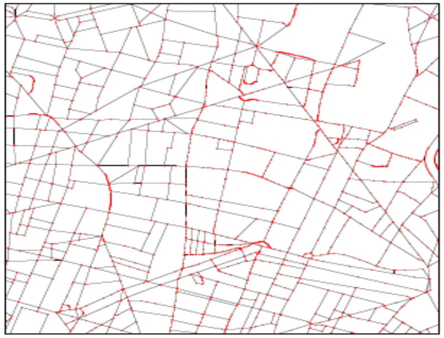
epidemics, forest fire, telecom grid, www
Nanonets, photovoltaics

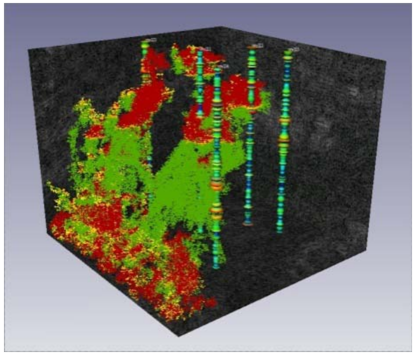
Cluster sizes

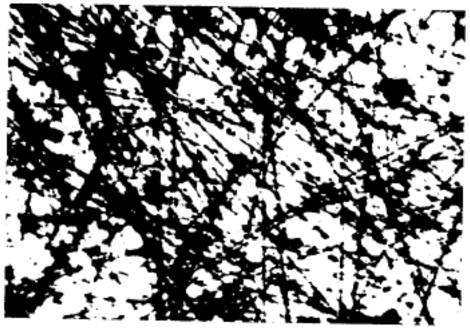
Oil fields, NC Flash

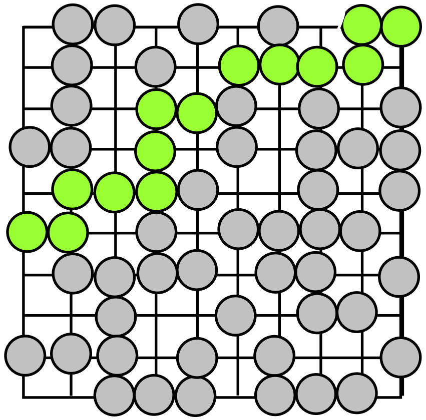
Fractal dimension

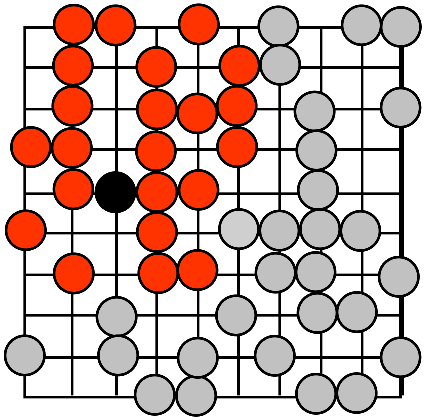
Aerosol, paper, sensors

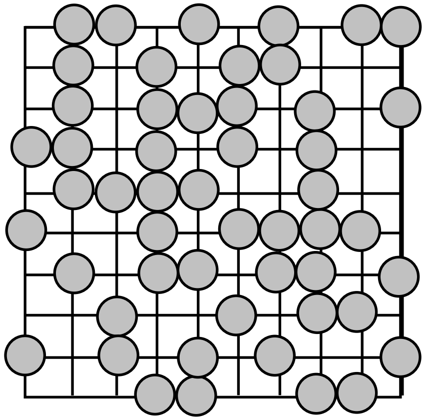








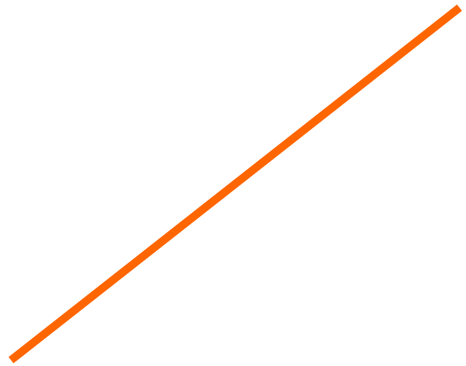




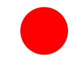
basic concepts: dimension of a surface

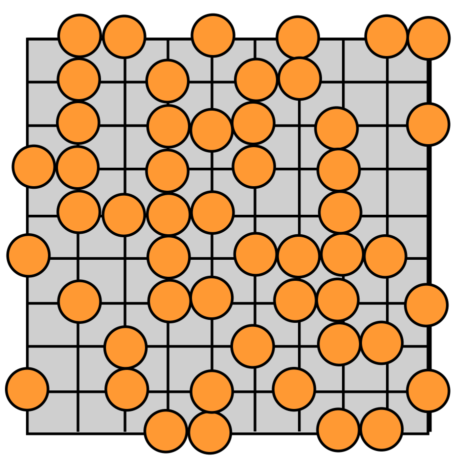
D=2

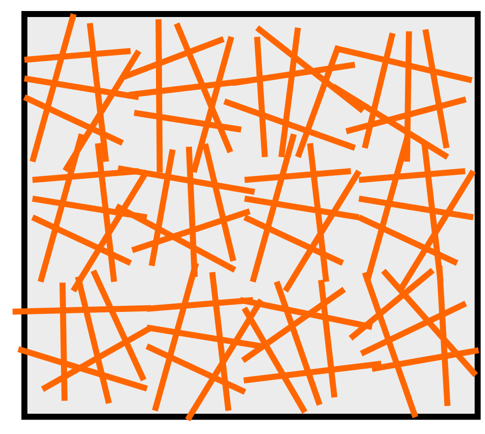
D=1



D=0



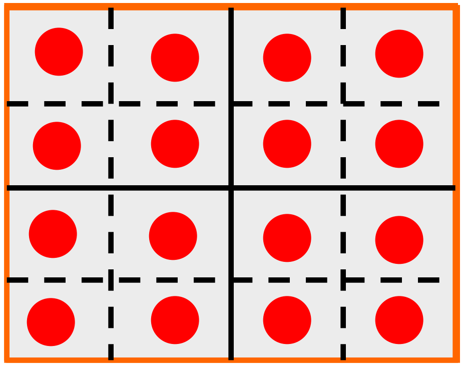




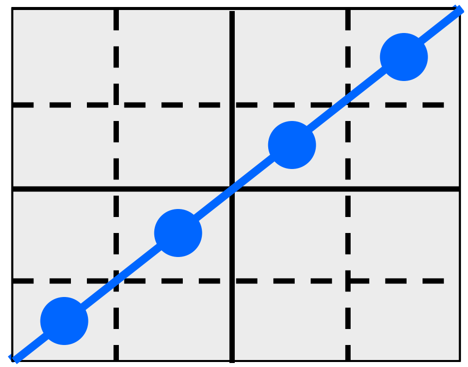
D=?

classification of surfaces…

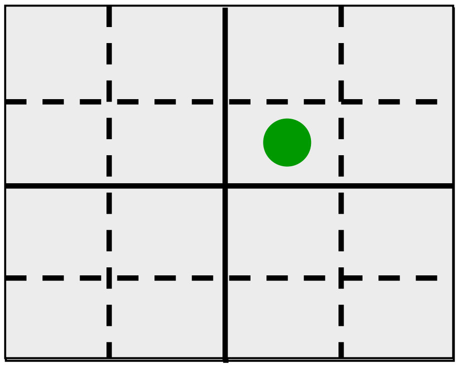
Fractal Dimension (D_F)- Box counting technique



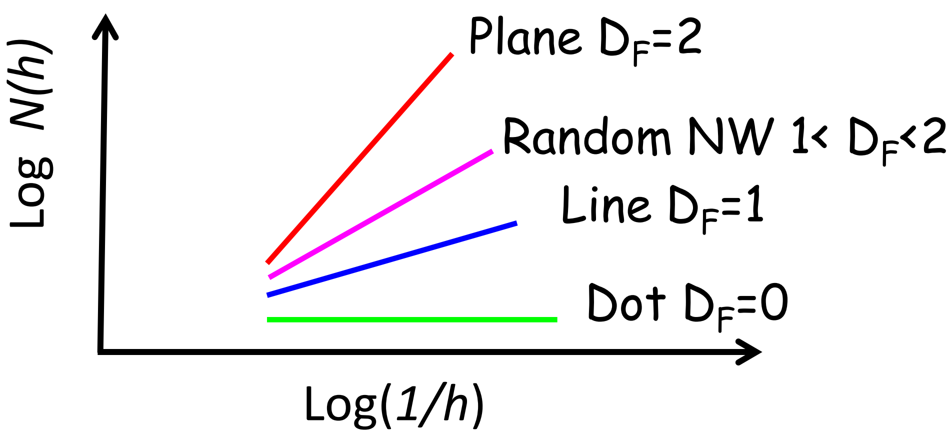
$N(h) \sim h^2$



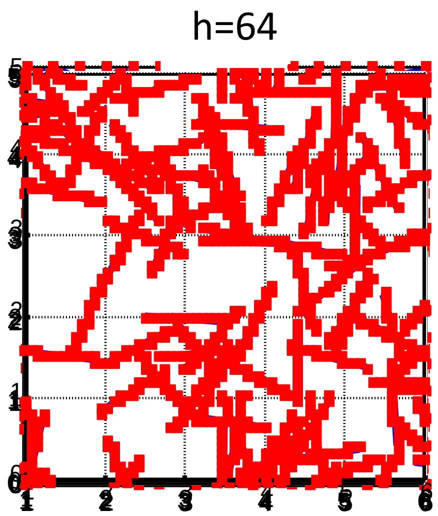
$N(h) \sim h^1$

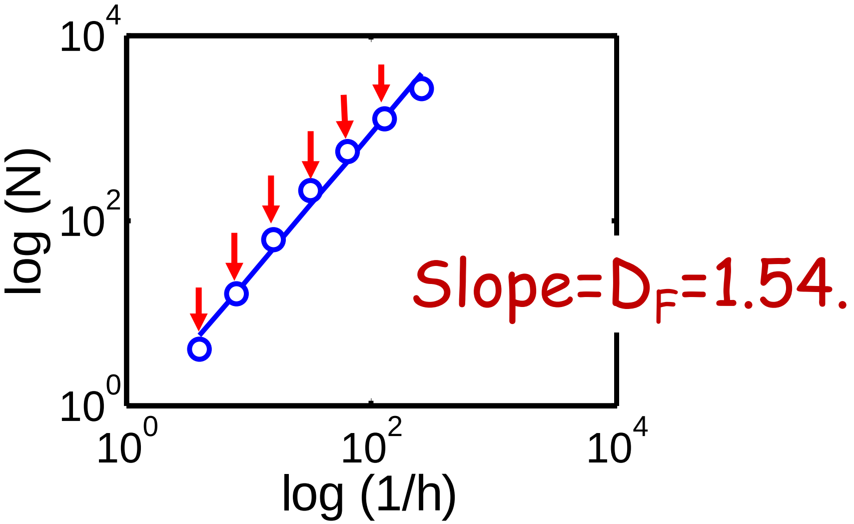


$N(h) \sim h^0$



example: dimension of a stick network

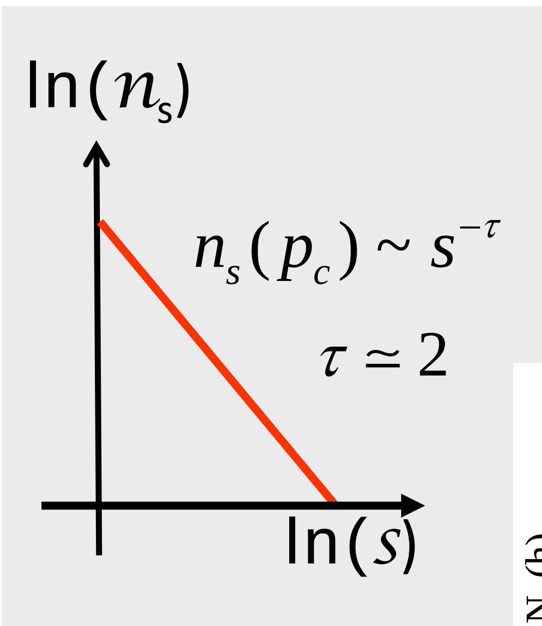


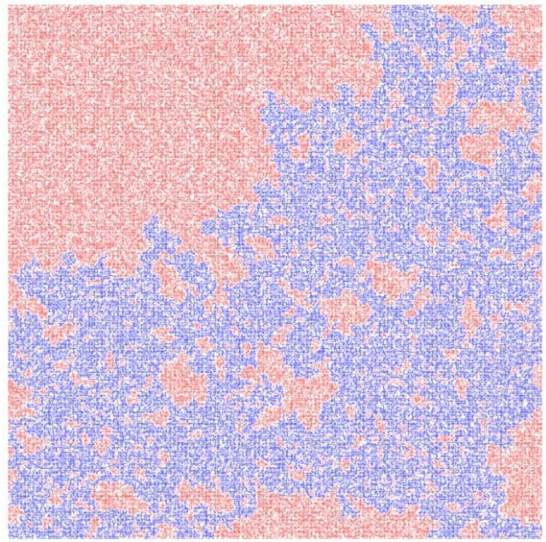


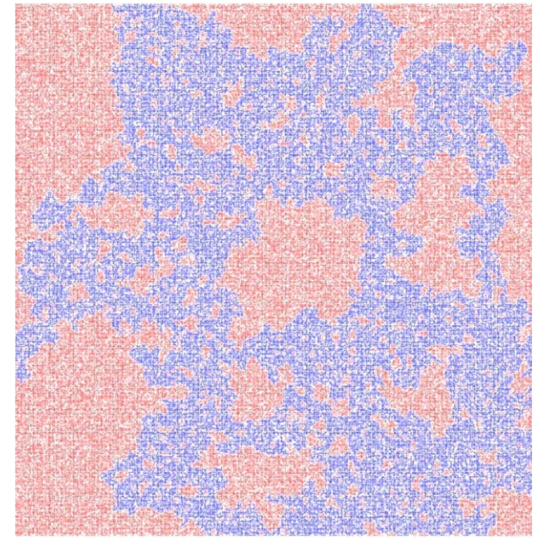
Dimension depends on stick density …

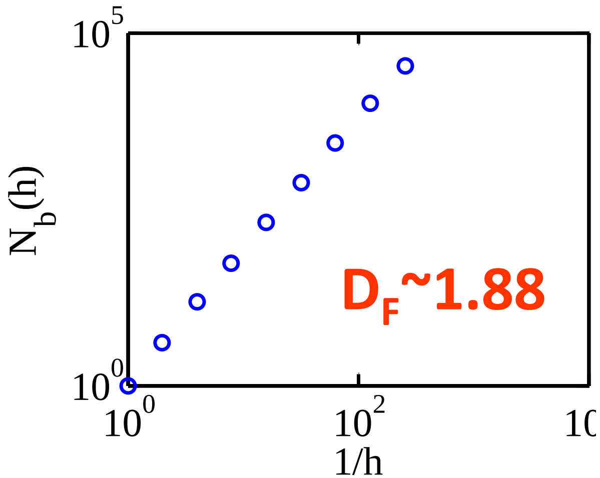
fractal dimension at percolation

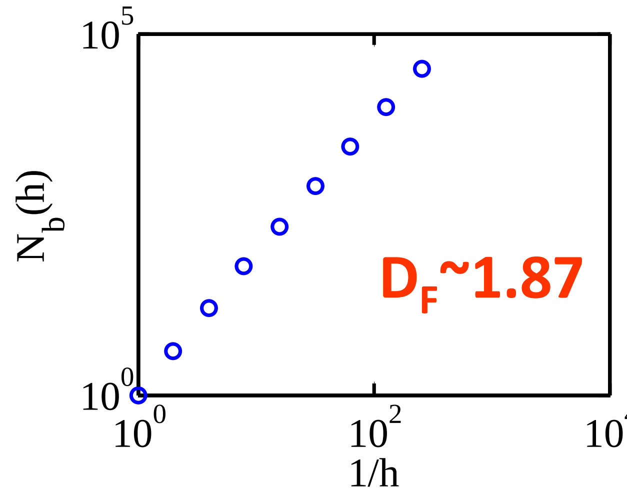
self-similarity











making of a fractal: dimension of Cantor dust

h	1/3
N	2

h	1/9
N	4

h	1/27
N	8

h	$1/3^n$
N	2^n

$$D_{F,1} = \frac{\log(N)}{\log(1/h)} = \frac{\log(2^n)}{\log(3^n)} = \frac{\log(2)}{\log(3)} = 0.63$$

Bigger than a point,
but smaller than line !

$$\text{In general,} \quad D_{F,1} = \frac{\log(m)}{\log(n)}$$

.... keep m piece

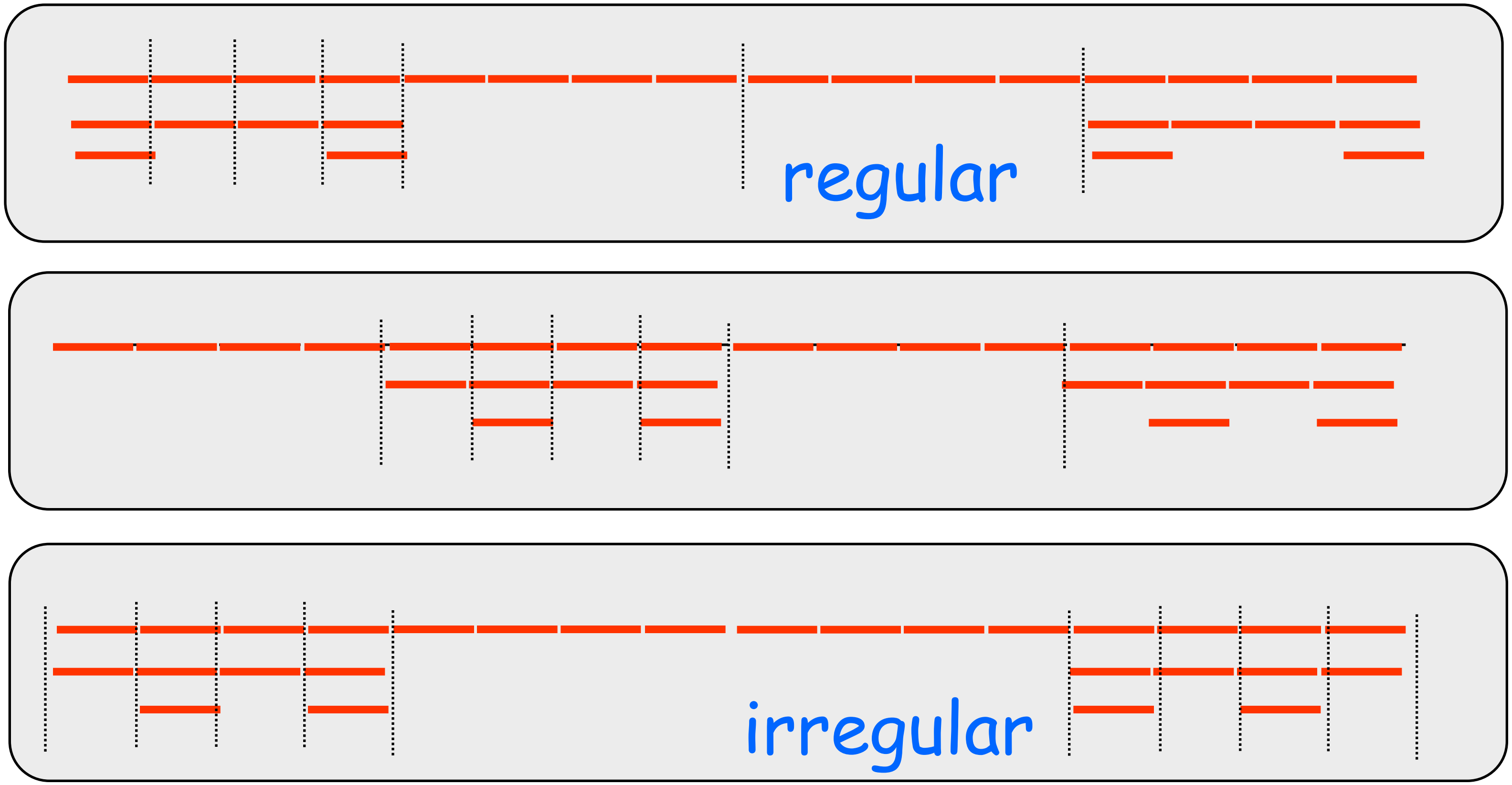
of n pieces

regular and irregular fractals

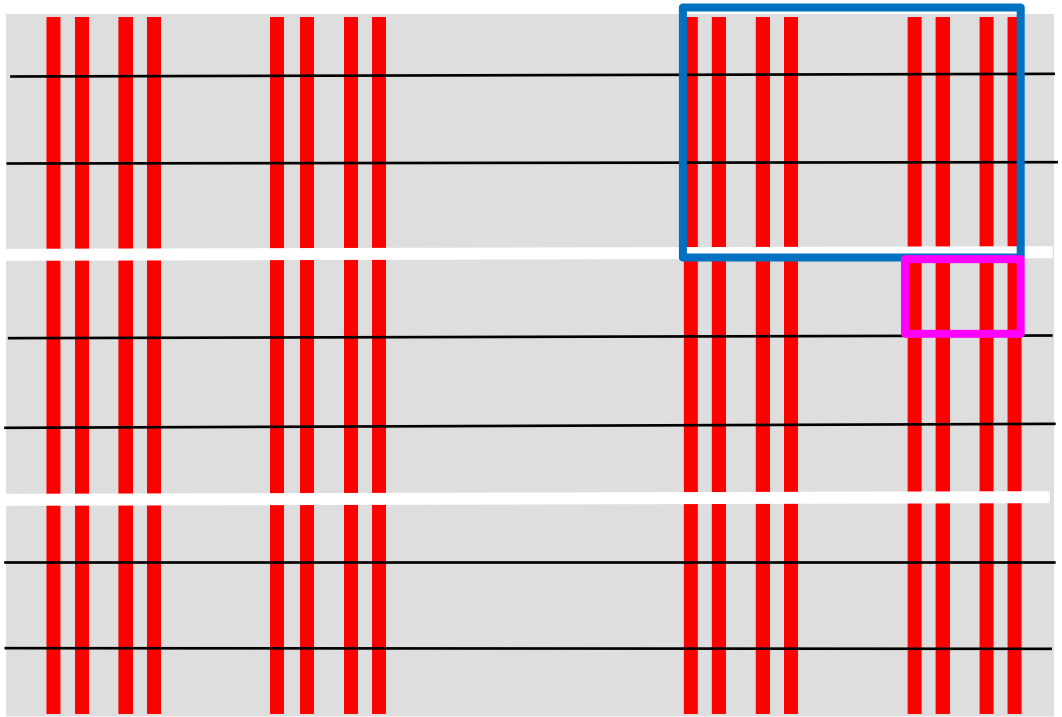
$$D_{F,1} = \frac{\log(m)}{\log(n)}$$

.... keep m piece

of n pieces



dimension of quasi-2D cantor stripes

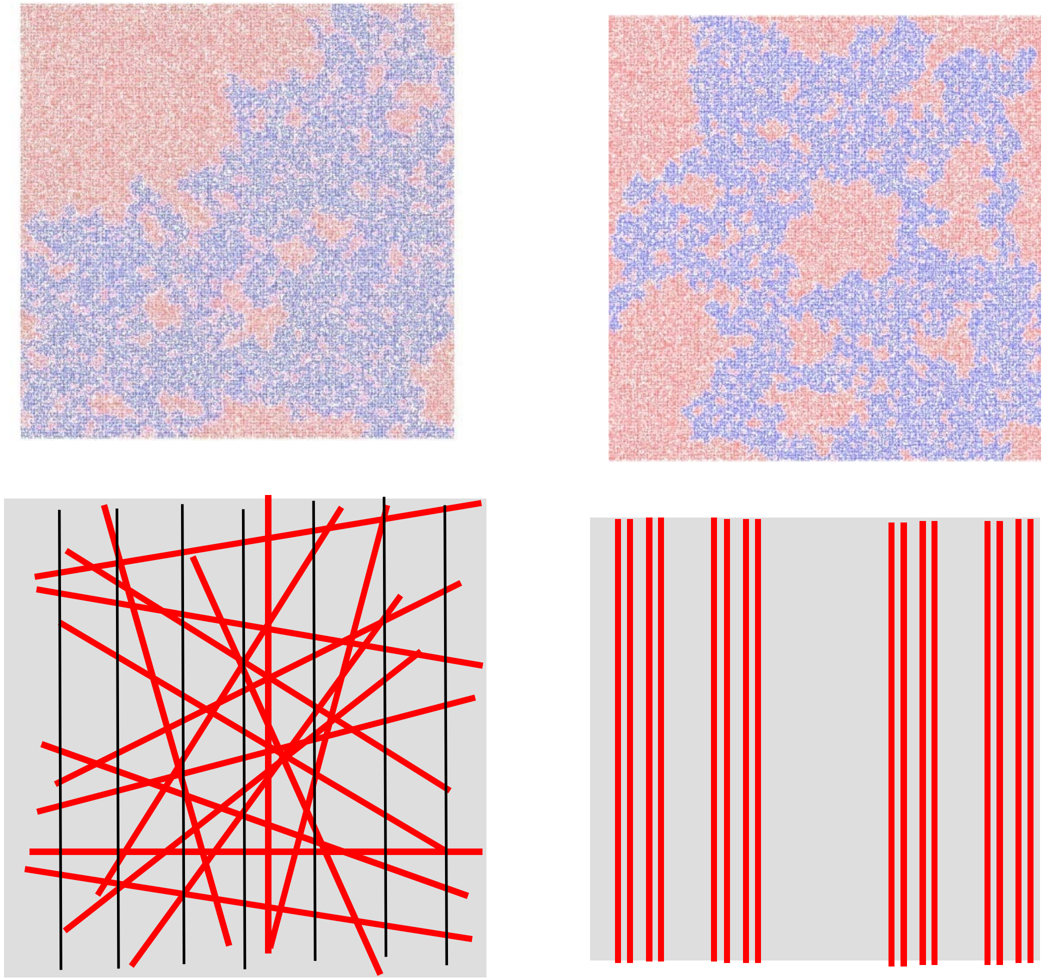


h	N
1/3	6
1/9	36
1/27	216
$1/3^n$	$3^n 2^n$

$$D_{F,2} = \frac{\log(N)}{\log(1/h)} = \frac{\log(3^n) + \log(2^n)}{\log(3^n)} = 1 + \frac{\log(2)}{\log(3)} = 1 + DF_x$$

In general, $D_{F,3} = DF_x + DF_y + DF_z$

cantor transform



Preserve D_F during transformation (Lecture 5)

conclusion

- Discussed three key concepts of percolative transport: percolation threshold, island size distribution, and fractal dimension

- The concept of excluded volume provides a (nearly) geometry independent way for calculating the percolation threshold for arbitrarily shaped objects on arbitrary grid.

- Distribution of island sizes is also described by simple formula with universal constants. At percolation threshold, the island sizes are self-similar and scale invariant.

- Fractal dimension provides a generalized technique to describe the dimension of any surfaces, even those defined by randomly oriented sticks. Cantor transform regularizes the structure.

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The discussion on excluded volume is based on I. **Balberg**, C. Anderson, S Alexander, N Wagner - Physical Review B, 1984.

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