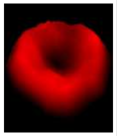

Optical Imaging
Chapter 3 – Imaging

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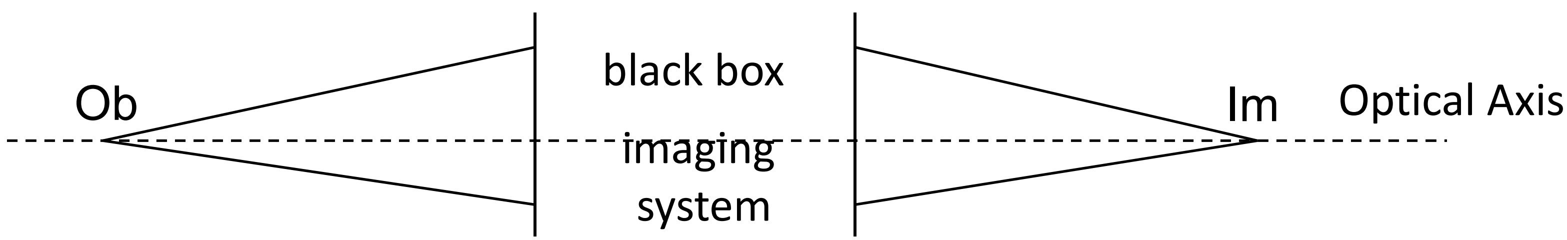
Objectives

- Introduction to geometrical optics and Fourier optics – precedes Microscopy

3.1 Geometrical Optics

- If the objects encountered by light are large compared to wavelength, the equations of propagation can be greatly simplified ($\lambda \rightarrow 0$)
- i.e. the wave-phenomena (scattering, interference, etc) are neglected
- In homogeneous media, light travels in straight lines ≡ rays
- G.O. deals with ray propagation trough optical media (eg. Imaging systems)

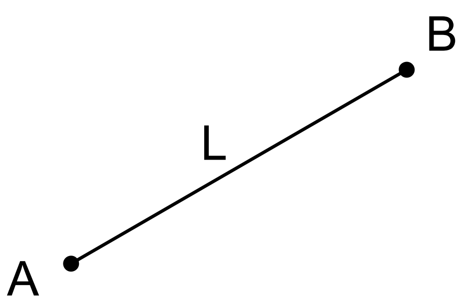


3.1 Geometrical Optics

- G.O. predicts image location trough complicated systems; accuracy is fairly good
- Nowadays there are software programs that can run “ray propagation” trough arbitrary materials
- So, what are the laws of G.O.?

3.2 Fermat's principle

a) n = constant

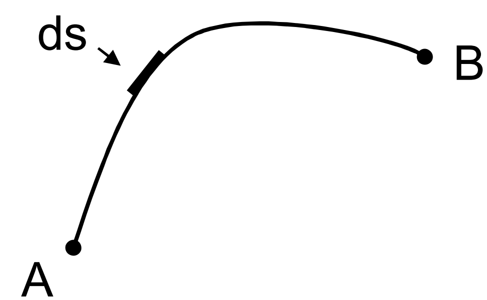


$$v = \frac{c}{n}$$

Time: $t_{AB} = \frac{L}{v} = \frac{1}{c} nL$

→ straight line

b) n = n($\bar{r}$) = function of position



$$v(r) = \frac{c}{n(r)}$$

$$dt = \frac{ds}{v_B} = \frac{1}{c} n(s) ds$$

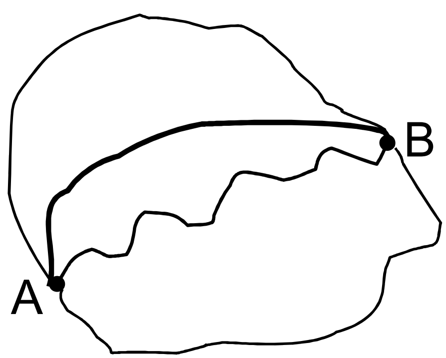
$$\rightarrow \boxed{t_{AB} = \frac{1}{c} \int_A n(s) ds} \qquad (3.1)$$

3.2 Fermat's principle

- Definition:

$S = ct = \int n(s)ds \equiv$ optical path length (3.2)

- How can we predict ray bending (eg. mirage)?
- <u>Fermat's Principle</u>:
  - Light connects any two points by a path of minimum time (the least time principle)

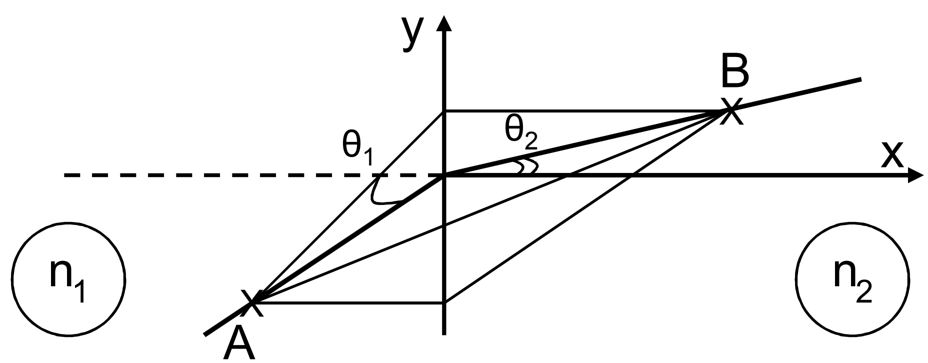


$$\delta\left[\int_A^B n(S)dS\right] = 0 \quad (3.3)$$

  - If n=constant in space, AB=line, of course

3.3 Snell's Law

- Consider an interface between 2 media:



- The rays are "bent" such that: (3.4)

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

- Snell's law (3.4) can be easily derived from Fermat's principle, by minimizing:

$$S = n_1 \left|AO\right| + n_2 \left|OB\right| = \text{total path-length}$$

- Take it as an exercise

3.3 Snell's Law

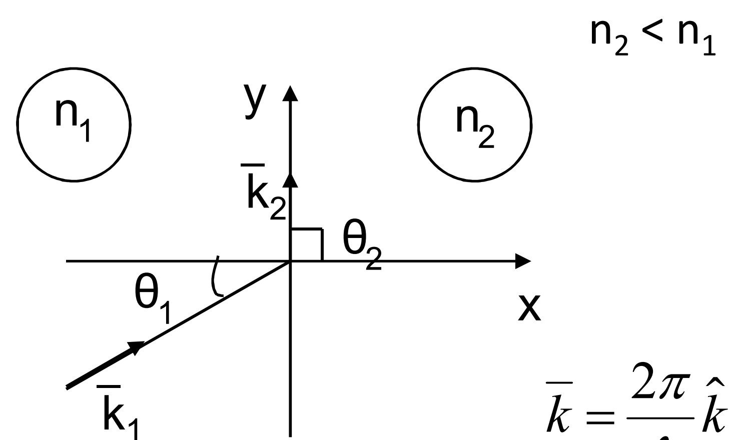
- Consequences of Snell's Law:

a) If $n_2 > n_1$ → $\Theta_2 < \Theta_1$ (ray gets closer to normal)

b) If $n_2 < n_1$ → quite interesting!

$$\theta_2 = \sin^{-1}\left[\frac{n_1}{n_2}\sin\theta_1\right] \qquad (3.5)$$

→ Ray gets away from normal



$$\bar{k} = \frac{2\pi}{\lambda}\hat{k}$$

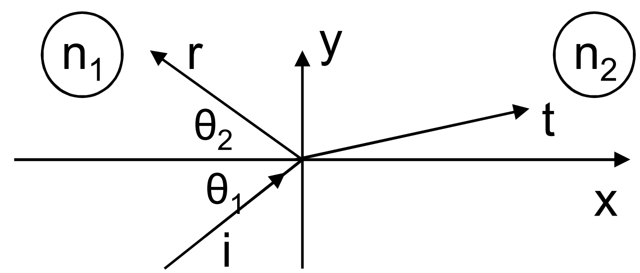
- So, if $\left[\frac{n_1}{n_2}\sin\theta_1\right] = 1 \Rightarrow \boxed{\theta_2 = \pi/2}$ NO TRANSMISSION

3.3 Snell's Law

- The angle of incidence θ_c for which

$$n_1 \sin\theta_c = n_2 \qquad (3.6)$$

  is called critical angle
- This is total internal reflection



c) law of reflection

$n_2 = -n_1$ Snell's law is:

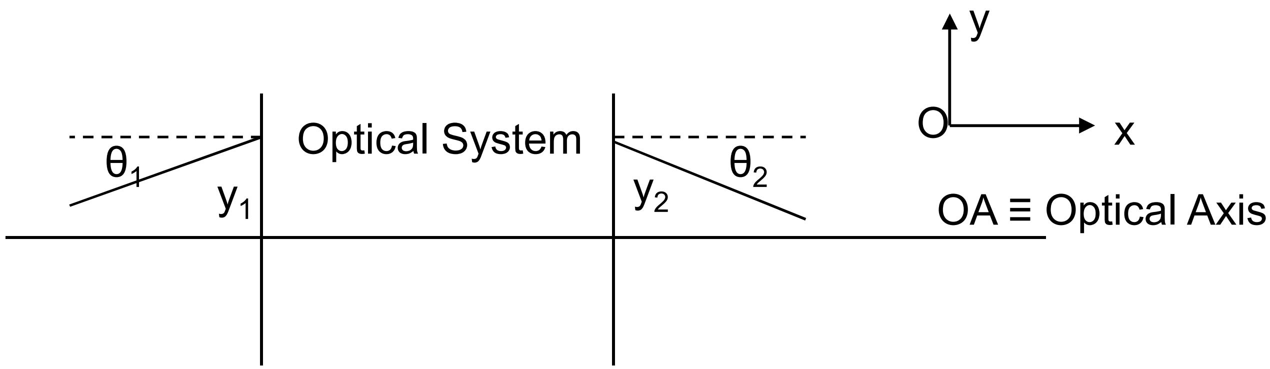
$$\rightarrow \quad n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$$\rightarrow \quad \theta_1 = -\theta_2 \text{ (reflection law)} \qquad (3.7)$$

- Energy conservation: $P_t + P_r = P_i$

3.4 Propagation Matrices in G.O

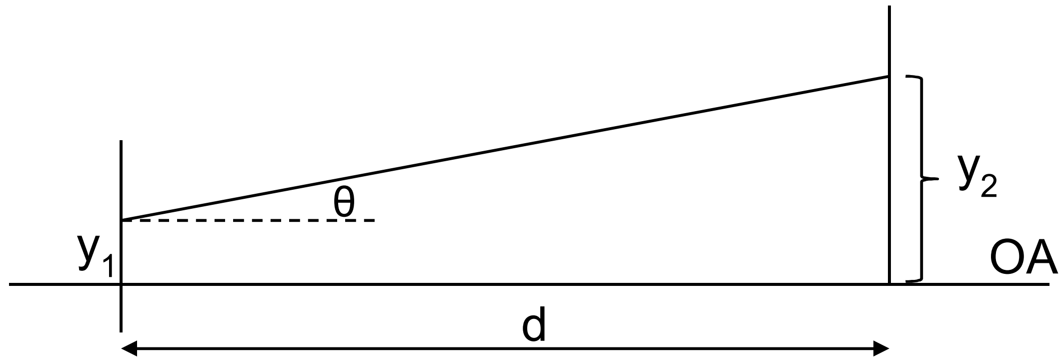
- Efficient way of propagating rays through optical systems



- Any given ray is completely determined at a certain plane by the angle with OA, Θ_1, and height w.r.t OA, y_1 ≡
- Let's propagate (y_1, Θ_1), assume small angles Gaussian approximation

3.4 Propagation Matrices in G.O

a) Translation



$$\begin{cases} \theta_2 = \theta_1 \\ y_2 = y_1 + d \tan\theta_1 \end{cases}$$

Small angles:

$$\begin{cases} y_2 = y_1 + d\theta_1 \\ \theta_2 = 0y_1 + 1\theta_1 \end{cases} \qquad (3.8)$$

3.4 Propagation Matrices in G.O

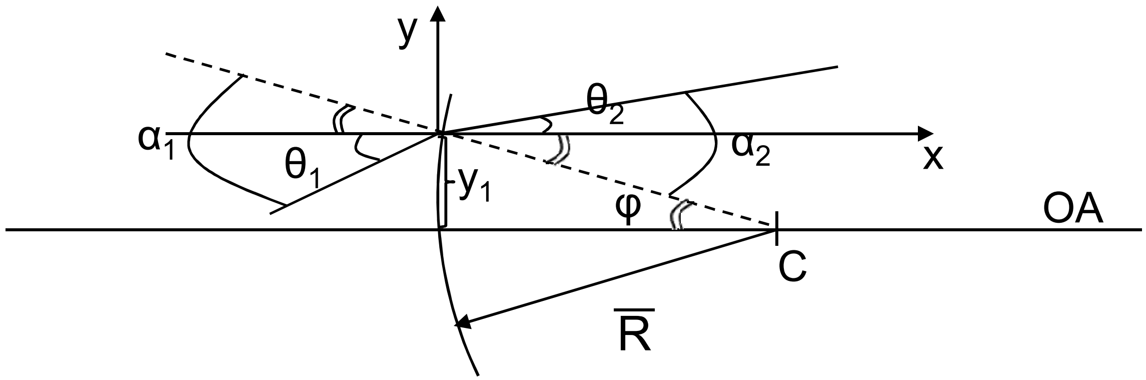
a) Translation

- We can re-write in compact form:

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix} \tag{3.9}$$

3.4 Propagation Matrices in G.O

b) Refraction-spherical dielectric interface



- Snell's law: $n_1\alpha_1 = n_2\alpha_2$
- Geometry: $\begin{cases} \alpha_1 = \theta_1 + f \\ \alpha_2 = \theta_2 + f \\ f = \dfrac{y_1}{R} = \dfrac{y_2}{R} \end{cases}$ (3.10)

$$\rightarrow n_1\theta_1 + \frac{n_1}{R}y_1 = n_2\theta_2 + \frac{n_2}{R}y_2 \,|\cdot \frac{1}{n}$$

3.4 Propagation Matrices in G.O

b) Refraction-spherical dieletric interface

- So: $\begin{cases} y_2 = y_1 + 0 \cdot \theta_1 \\ \theta_2 = (\frac{n_1}{n_2} - 1)\frac{y_1}{R} + \frac{n_1}{n_2}\theta_1 \end{cases}$

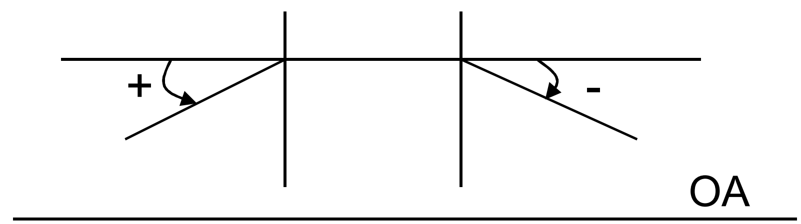
$$\rightarrow \quad \begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{n_1 - n_2}{n_2 R} & \frac{n_1}{n_2} \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix} \qquad (3.11)$$

3.4 Propagation Matrices in G.O

b) Refraction-spherical dieletric interface

- Important: To avoid confusion between Ɵ and – Ɵ angles, use "sign convention"

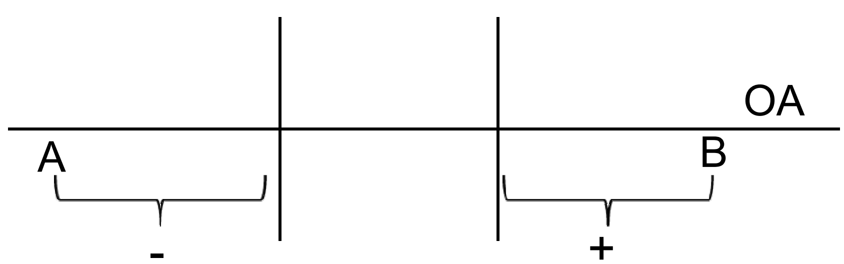
1. angle convention



- Counter clock-wise = positive

2. distance convention

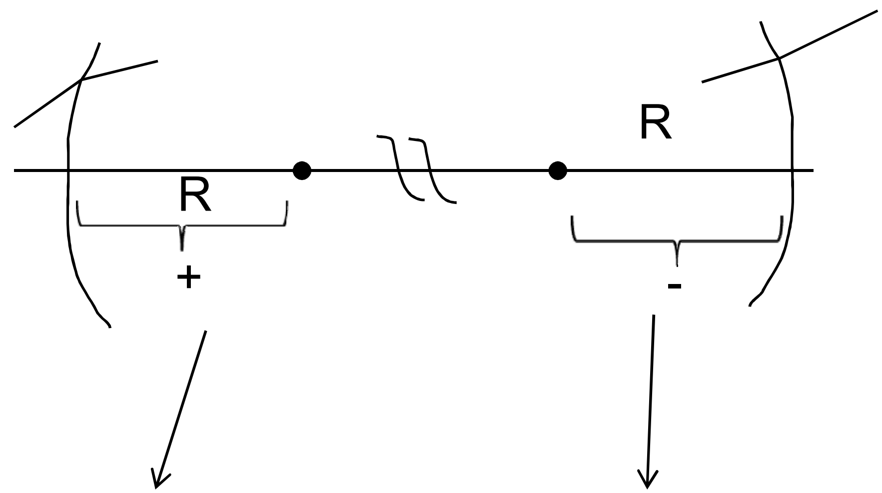
Left → negative
Right → positive



3.4 Propagation Matrices in G.O

b) Refraction-spherical dieletric interface

- Example:

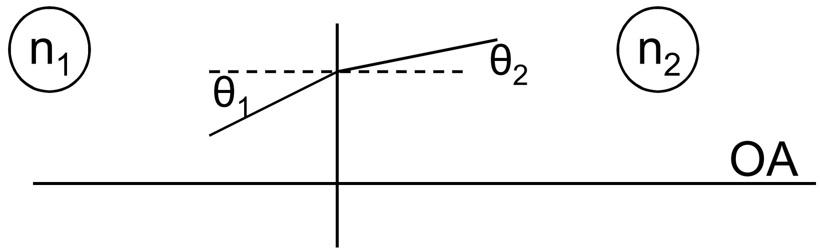


We found $\begin{pmatrix} 1 & 0 \\ \frac{n_1 - n_2}{n_2 R} & \frac{n_1}{n_2} \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \frac{n_2 - n_1}{n_2 R} & \frac{n_1}{n_2} \end{pmatrix}$

Same +/- convention applies to spherical mirrors. Without sign convention, it's easy to get the wrong numbers.

3.4 Propagation Matrices in G.O

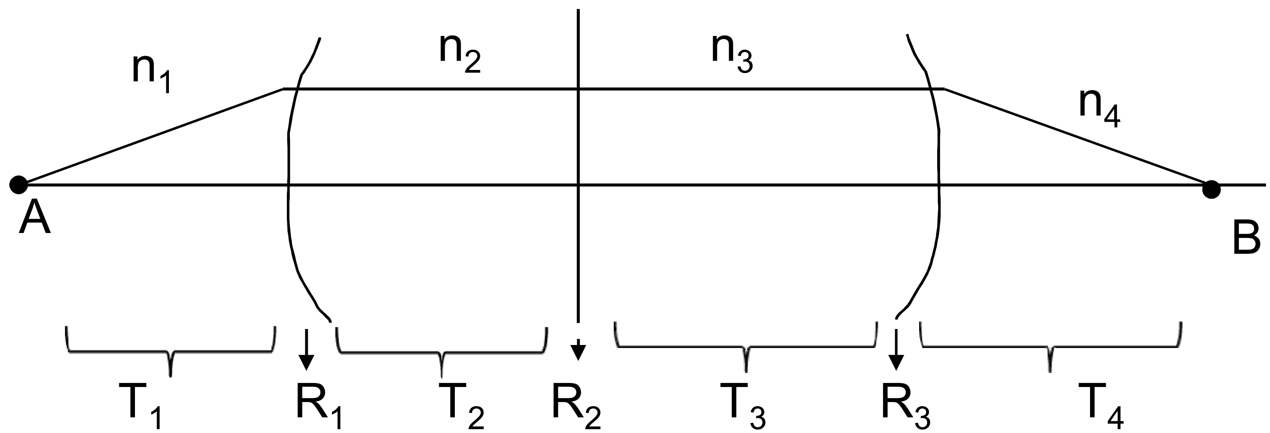
c) Dielectric interface – particular case of R→ ∞



$$\lim_{R\to\infty}\begin{pmatrix} 1 & 0 \\ \frac{n_1-n_2}{n_2 R} & \frac{n_1}{n_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix} \qquad (3.12)$$

3.4 Propagation Matrices in G.O

- The nice thing is that cascading multiple optical components reduces to multiplying matrices (linear systems)
- Example:



$$\begin{pmatrix} y_B \\ \theta_B \end{pmatrix} = T_4 \cdot R_3 \cdot T_3 \cdot R_2 \cdot T_2 \cdot R_1 \cdot T_1 \cdot \begin{pmatrix} y_A \\ \theta_A \end{pmatrix} \quad (3.13)$$

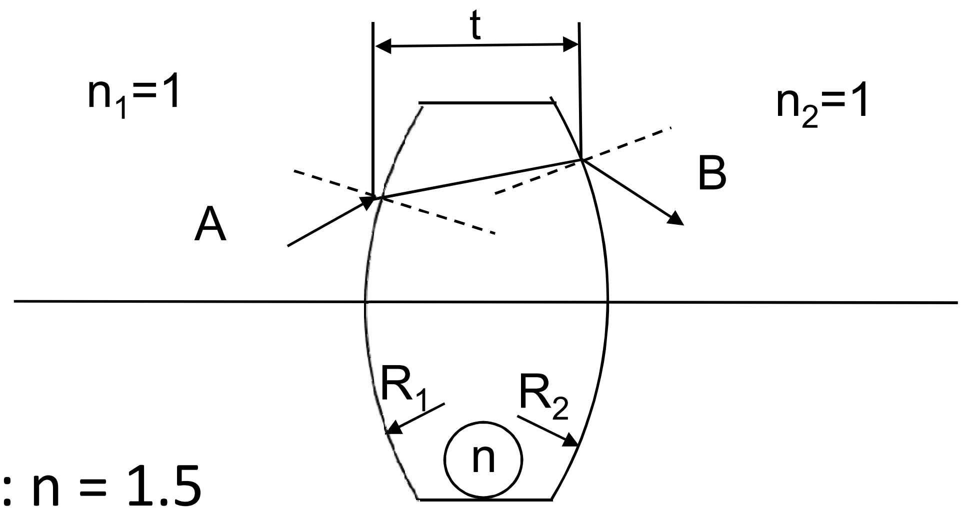
- Note the reverse order multiplication (chronological order)

3.4 Propagation Matrices in G.O

- Note the reverse order multiplication (chronological order)
- T = Translation matrix = $\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$

- R=refraction matrix = $\begin{pmatrix} 1 & 0 \\ \dfrac{n_2 - n_1}{n_2 R} & \dfrac{n_1}{n_2} \end{pmatrix}$

3.5 The thick Lens



- Typical glass: n = 1.5
- Basic optical component: typically - 2 spherical surfaces

$$\begin{pmatrix} y_B \\ \theta_B \end{pmatrix} = R_B \cdot T_t \cdot R_A . \begin{pmatrix} y_A \\ \theta_A \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 \\ \frac{n-1}{R_2} & n \end{pmatrix} \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{1-n}{nR_1} & \frac{1}{n} \end{pmatrix}}_{M} . \begin{pmatrix} y_A \\ \theta_A \end{pmatrix}$$

3.5 The thick Lens

- After some algebra:

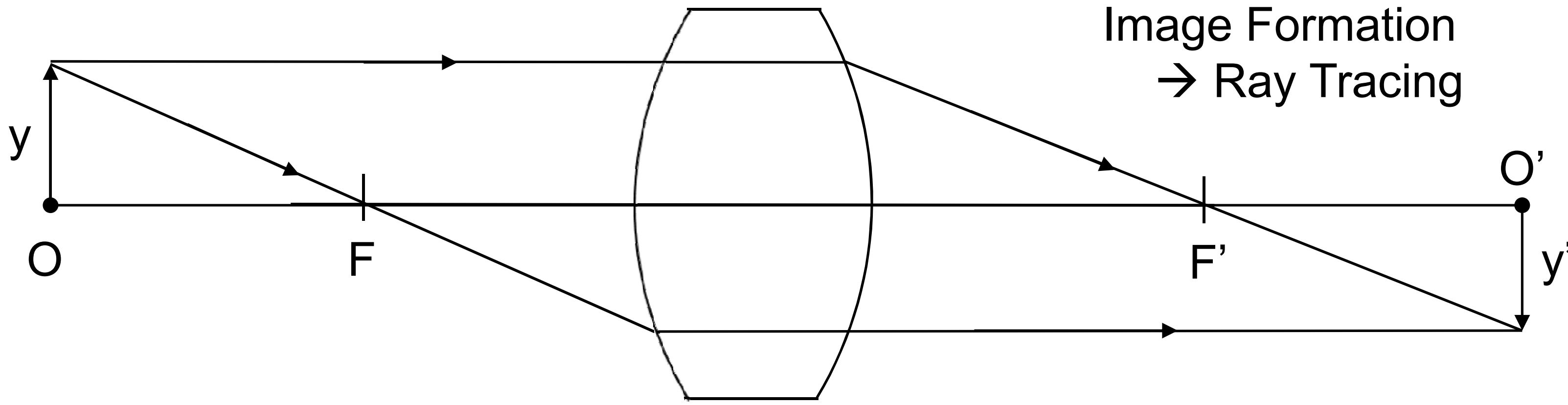
$$M = \begin{pmatrix} 1 - C_1 \frac{t}{R_1} & \frac{t}{n} \\ -(C_1 + C_2 - C_1 C_2 \frac{t}{n}) & 1 - C_2 \frac{t}{nR_2} \end{pmatrix} \qquad (3.14)$$

- In general $C = \frac{n_2 - n_1}{R} \equiv$ convergence of spherical surface
- $R_1 > 0$, $R_2 < 0$ → $C_1 > 0$ & $C_2 > 0$ → <u>convergent</u>
- Note $[C] = m^{-1}$ = dioptries

3.5 The thick Lens

- Definition: $$\frac{1}{f} = C_1 + C_2 - C_1 C_2 \frac{t}{n} \qquad (3.15)$$
- f is the focal distance of lens
- Eq (3.15) is the "lens makers equation"

3.6 Cardinal points

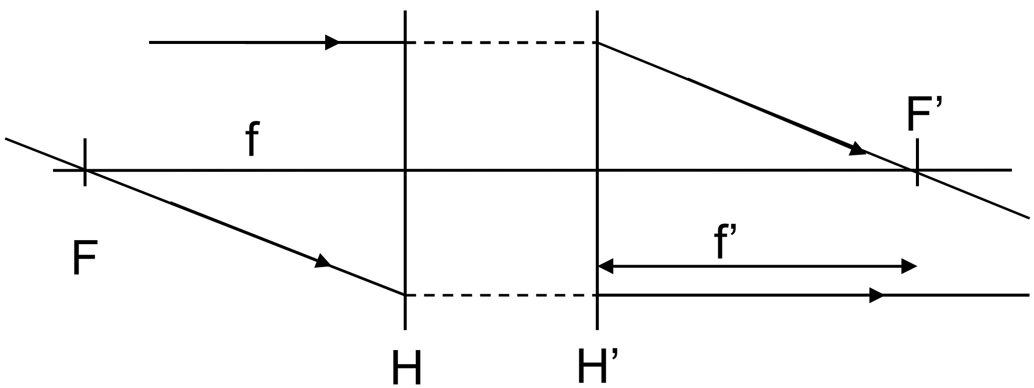


- O = object; O'=image ; O-O'=conjugate points
- F' = focal point image (image of objects from -∞)
- F = focal point object
- Transverse magnification:

$$M = \frac{y'}{y} \qquad (3.16)$$

3.6 Cardinal points

- Definition: principal planes are the conjugate planes for which M = 1



H, H' = principal planes

f, f' = focal distances

! f, f' measured from H

3.7 Thin lens

- Particular use: t → 0
- Transfer matrix for thin lens:

$$\lim_{t\to 0}\begin{pmatrix} 1-C_1\dfrac{t}{R_1} & \dfrac{t}{n} \\ -(C_1+C_2-C_1C_2\dfrac{t}{n}) & 1-C_2\dfrac{t}{nR_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -(C_1+C_2) & 1 \end{pmatrix}$$

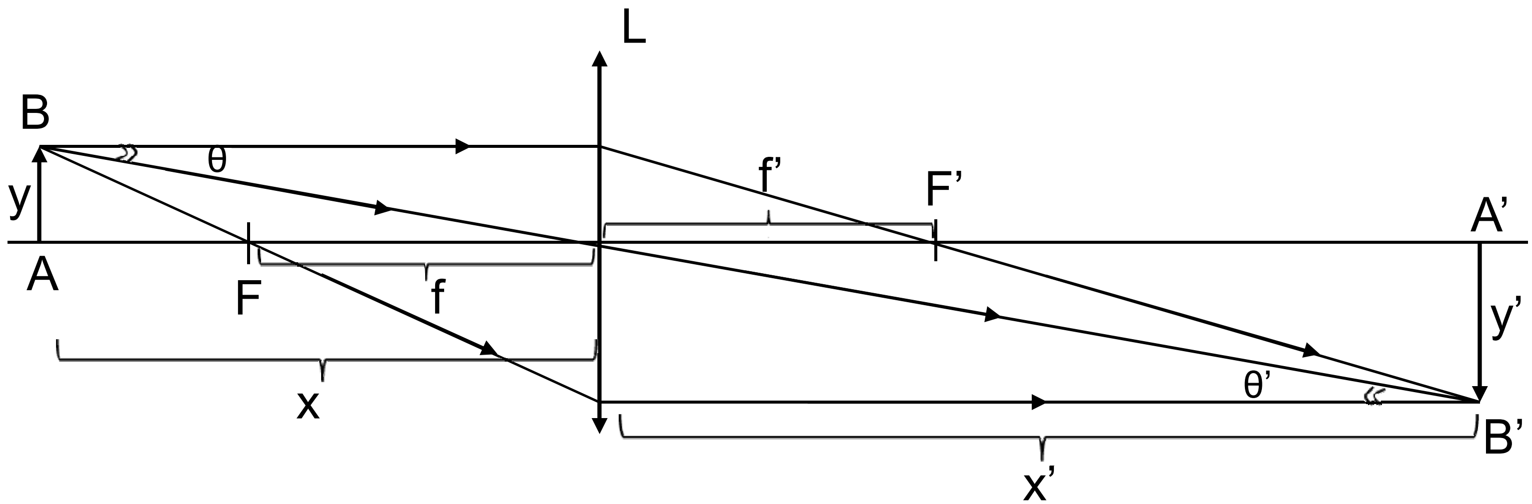
- Since $\dfrac{1}{f} = C_1 + C_2 = (n-1)(\dfrac{1}{R_1} - \dfrac{1}{R_2})$
- (Note $R_1 > 0$, $R_2 < 0$)

→ $M_{\text{thin lens}} = \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{pmatrix}$ (3.17)

3.7 Thin lens

- Remember other matrices:
  - Translation: $$T = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \quad (3.18)$$
  - Refraction-spherical surface: $$R = \begin{pmatrix} 1 & 0 \\ \dfrac{n_2 - n_1}{n_2 R} & \dfrac{n_1}{n_2} \end{pmatrix} \quad (3.19)$$
  - Spherical mirror: $$M = \begin{pmatrix} 1 & 0 \\ -\dfrac{2}{R} & 1 \end{pmatrix} \quad (3.20)$$
    (f = R/2)

3.8 Ray Tracing – thin lenses



-  = <u>convergent</u> lens; $f > 0$
-  = <u>divergent</u> lens; $f < 0$

3.8 Ray Tracing – thin lenses

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = T_{x'} M_f T_x \begin{pmatrix} y \\ \theta \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix} =$$

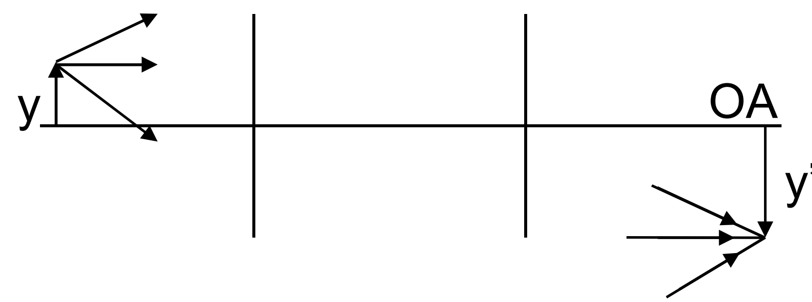
$$= \begin{pmatrix} 1 - \frac{x'}{f} & x + x' - \frac{xx'}{f} \\ -\frac{1}{f} & 1 - \frac{x}{f} \end{pmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix} \qquad (3.21)$$

3.8 Ray Tracing – thin lenses

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$; y' can be found as:

$$\boxed{y' = Ay + B\theta} \quad (3.22)$$

- Condition for conjugate planes:



- For conjugate planes, y' should be independent of angle ϴ
  → B = 0
- i.e. stigmatism condition (points are imaged into points)
- We neglect geometric/chromatic aberrations

3.8 Ray Tracing – thin lenses

- So, B = 0 → $x + x' - \frac{xx'}{f} = 0$

  → $$\boxed{\frac{1}{x'} + \frac{1}{x} = \frac{1}{f}} \qquad (3.23)$$

- Eq above is the conjugate points equation (thin lens)
- Eq 3.22 becomes: y'= yA

→ $M = A = 1 - \frac{x'}{f} =$ Transverse magnification (3.24)

3.8 Ray Tracing – thin lenses

- Use Eq 3.23:

$$M = 1 - \frac{x'}{f} = 1 - x'\left(\frac{1}{x'} + \frac{1}{x}\right)$$

$$= \frac{x'}{x} < 0 \quad \text{(inverted image)}$$

$$\boxed{M = \frac{x'}{x}} \qquad (3.25)$$

- If object and image space have different refractive indices, 3.23 has the more general form:

$$\frac{n'}{x'} + \frac{n}{x} = \frac{1}{f} \qquad (3.26)$$

3.8 Ray Tracing – thin lenses

- $x \to \infty \Rightarrow x' = n' f$
- $x' \to \infty \Rightarrow x = nf$

| f = focal distance in air

- Let's differentiate (3.26) for air, n'=n=1:

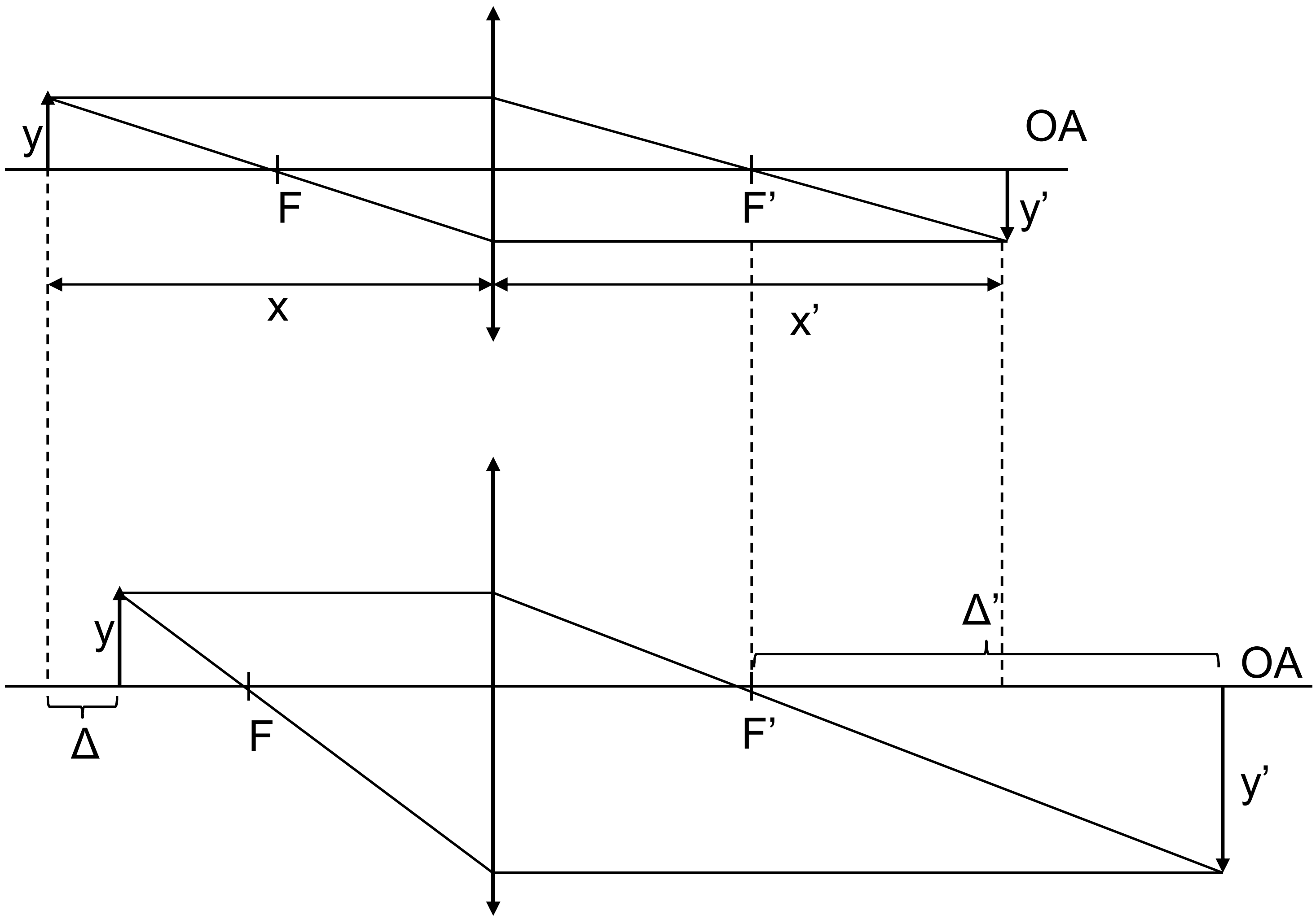
$$\frac{dx'}{x'^2} = -\frac{dx}{x^2}$$

$$dx' = -\left(\frac{x'}{x}\right)^2 dx$$

$$\boxed{dx' = -M^2 dx} \qquad (3.27)$$

- Eq 3.27 says that if the object gets closer to lens, the image moves away!

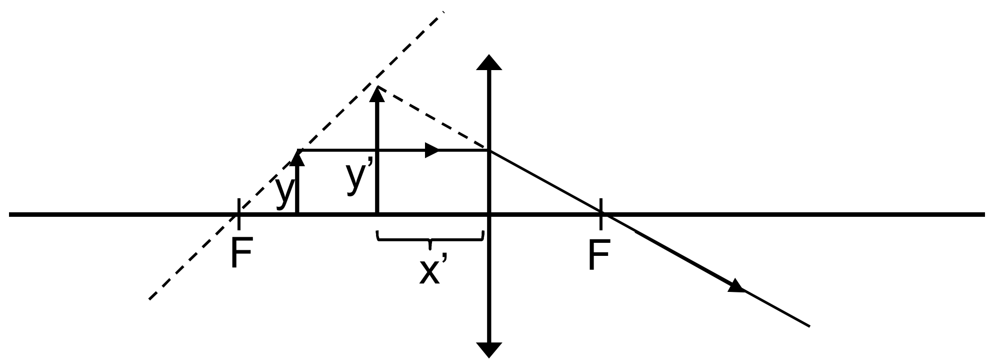
3.8 Ray Tracing – thin lenses



3.8 Ray Tracing – thin lenses

- What happens when x < f ?

$$\frac{1}{x'}+\frac{1}{x}=\frac{1}{f}\Rightarrow \boxed{\frac{1}{x'}=\frac{1}{f}-\frac{1}{x}<0}\ ?$$



- This image is formed by continuations of rays
  - Sometimes called "virtual images"
  - These images cannot be recorded directly (need re-imaging)

3.8 Ray Tracing – thin lenses

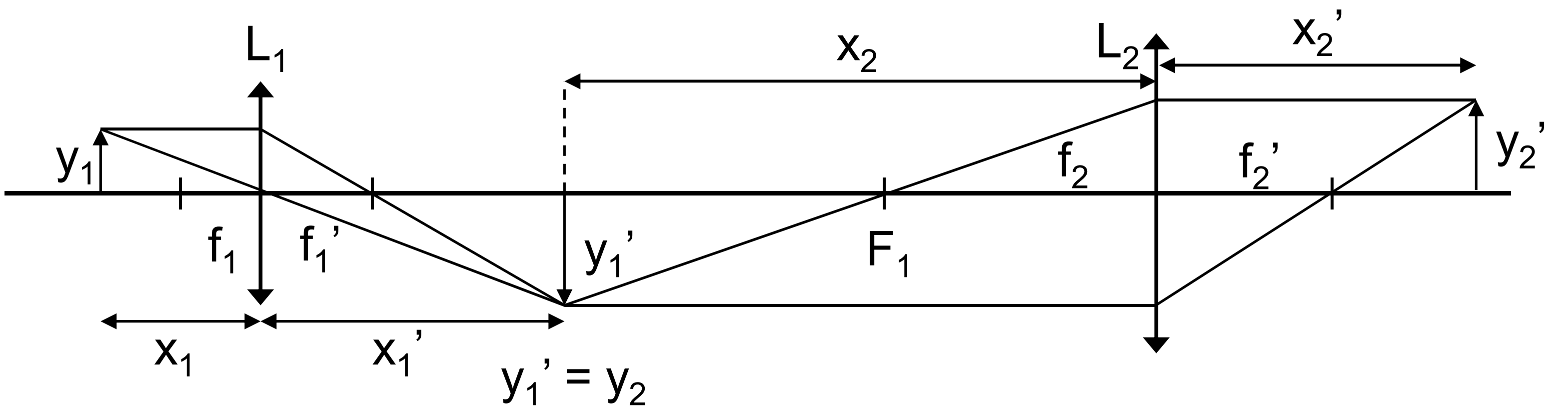
- Other useful formulas in G.O (figure above: Δ, Δ')
  - $\Delta.\Delta' = f^2$ (Newton's formula)

(3.28)

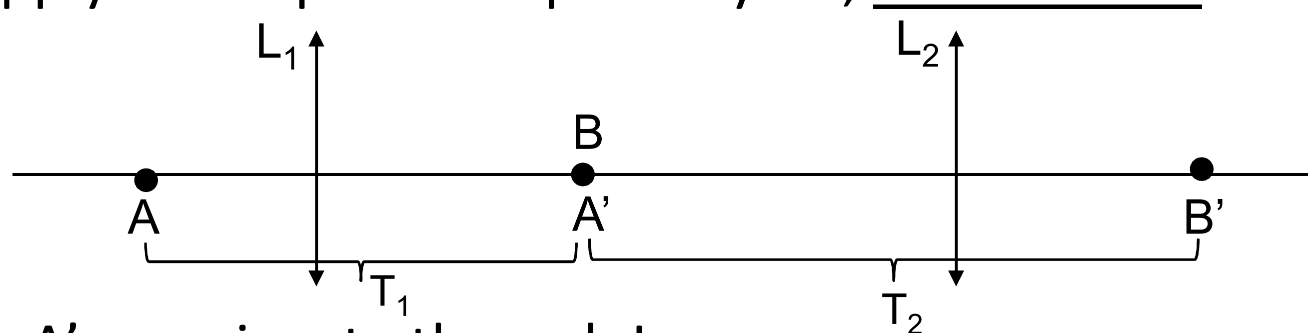
  - $\frac{y'}{y} = \frac{\Delta'}{f} = \frac{f}{\Delta}$ ("lens formula")

3.9 System of lenses

- The image through one lens becomes object for the next lens, etc



- Apply lens equation repeteatly. Or, <u>use matrices</u>



- A, A' = conjugate through L_1
- B,B' = conjugate through L_2

3.9 System of lenses

- Use T = $T_2.T_1$; T matrix from 3.21

$$T_1 = \begin{pmatrix} 1-\dfrac{x_1'}{f_1} & 0 \\ -\dfrac{1}{f_1} & 1-\dfrac{x}{f} \end{pmatrix} \qquad (3.29)$$

- Note: $1-\dfrac{x_1}{f_1} = 1 - x_1\left(\dfrac{1}{x_1}+\dfrac{1}{x_1'}\right) =$

$$= 1 - 1 - \frac{x_1}{x_1'} = \frac{1}{M_1} = \text{ magnification}$$

3.9 System of lenses

$$\rightarrow \begin{cases} 1-\dfrac{x_1}{f_1}=\dfrac{1}{M_1} \\ 1-\dfrac{x_1'}{f_1}=M_1 \end{cases} \qquad (3.30)$$

also:

! det(T_1) = 1

$$\rightarrow T = T_2 . T_1 =$$

$$= \begin{pmatrix} M_2 & 0 \\ -\dfrac{1}{f_2} & \dfrac{1}{M_2} \end{pmatrix} \begin{pmatrix} M_1 & 0 \\ -\dfrac{1}{f_1} & \dfrac{1}{M_1} \end{pmatrix} =$$

$$= \begin{pmatrix} M_1 M_2 & 0 \\ -\dfrac{M_1}{f_2}-\dfrac{1}{f_1 M_2} & \dfrac{1}{M_1 M_2} \end{pmatrix} \qquad (3.31)$$

3.9 System of lenses

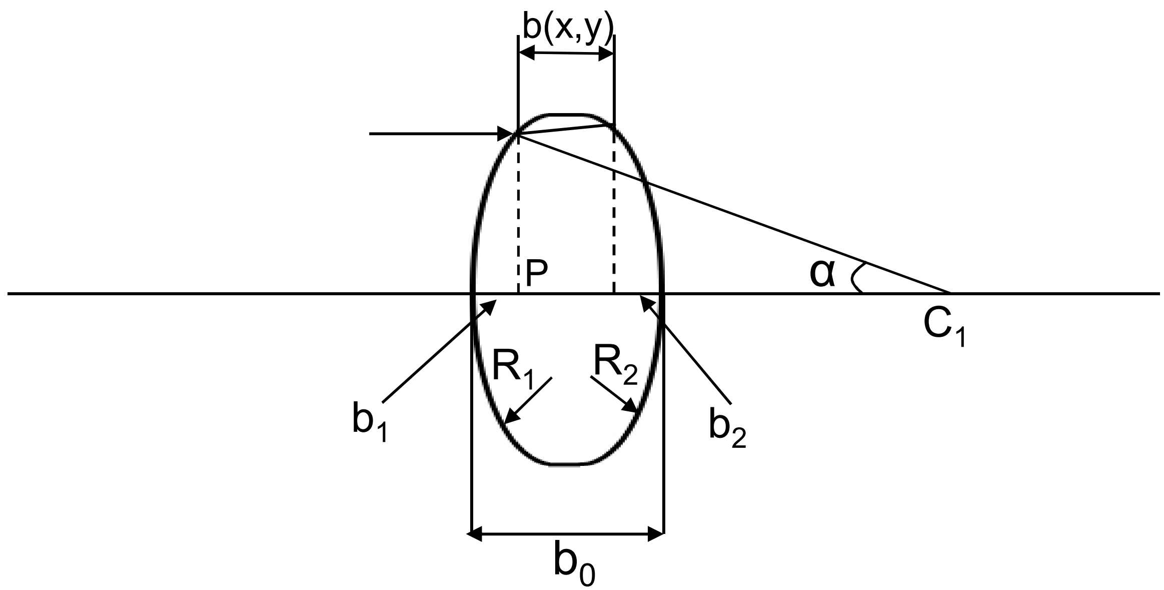
- 2-lens system is equivalent to:

$$\begin{cases} \frac{1}{f} = \frac{M_1}{f_2} + \frac{1}{f_1 M_2} \\ M = M_1 \cdot M_2 \end{cases}$$

- Microscopes achieve M=10-100 easily
- Can be reduced to 2-lens system
- Question: cascading many lenses such that $M=10^6$, would we be able to see atoms?
- Well, G.O can't answer that.
- So,back to wave optics

3.10 Lens as a phase transformer

- $E = E_o \cdot e^{i\phi}$ becomes important
- How is the wavefront changed in the vicinity of a lens?



- → $\phi = \underbrace{knb(x,y)}_{\text{glass}} + \underbrace{k[b_o - b(x,y)]}_{\text{air}}$ (3.32)

$$= kb_o + k(n-1)b(x,y)$$

3.10 Lens as a phase transformer

- Let's calculate b(x,y); assume small angles

$$b_1 = R_1 - (PC_1) =$$

$$= R_1 - \sqrt{R_1^{\ 2} - (\alpha R_1)^2} = R_1\left[1 - \sqrt{1-\alpha^2}\right]$$

- Taylor expansion: $\sqrt{1+x}\,|_{x\to o} \simeq 1 + \frac{x}{2}$

$$\rightarrow b_1 = R_1\left[1 - \left(1 - \frac{\alpha^2}{2}\right)\right] = R_1\frac{\alpha^2}{2} \qquad (3.33)$$

- $\alpha \simeq \tan\alpha = \dfrac{\sqrt{x^2+y^2}}{R_1}$

- So: $\boxed{b_1(x,y) = \dfrac{x^2+y^2}{2R_1}}$ (3.34)

3.10 Lens as a phase transformer

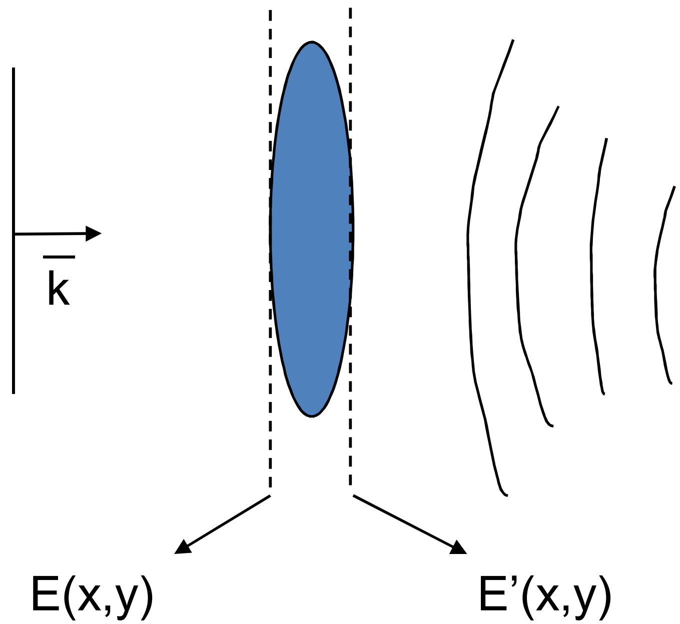
$$\rightarrow b(x,y) = b_o - b_1(x,y) - b_2(x,y) =$$
$$= b_o - \frac{x^2+y^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad (3.35)$$

- This is the thickness approximation
- The phase ϕ becomes:

$$\phi(x,y) = \phi_o - k(n-1)b(x,y) =$$
$$= \phi_o - k\frac{x^2+y^2}{2}(n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] \quad (3.36)$$

- But we know: $\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

3.10 Lens as a phase transformer



$$\rightarrow E'(x,y) = E(x,y) \cdot t_e(x,y) \qquad (3.37)$$

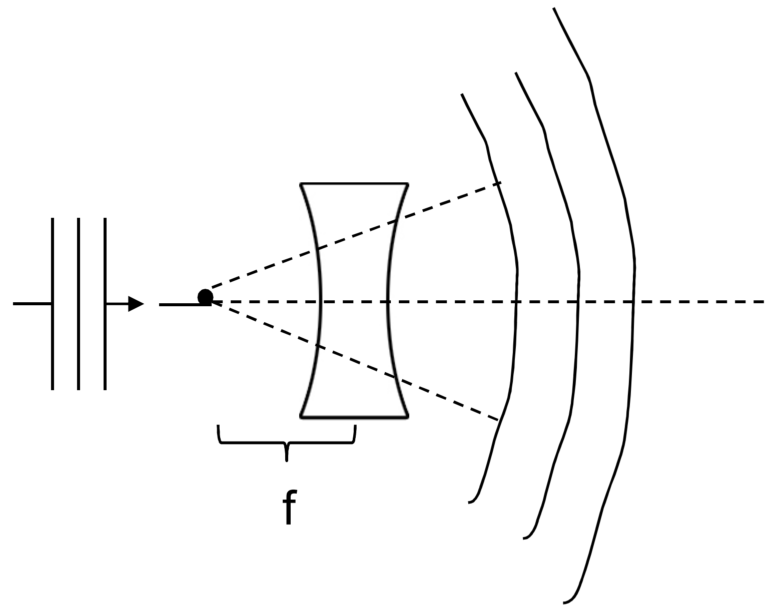
- The lens transformation is:

$$t_e = e^{i\phi} = \\ = e^{iknb_o} . e^{-i\frac{k}{2f}(x^2+y^2)} \qquad (3.38)$$

3.10 Lens as a phase transformer

- A lens transforms an incident plane wavefront into a parabolic shape
- Note: $f > 0$ covergent lens

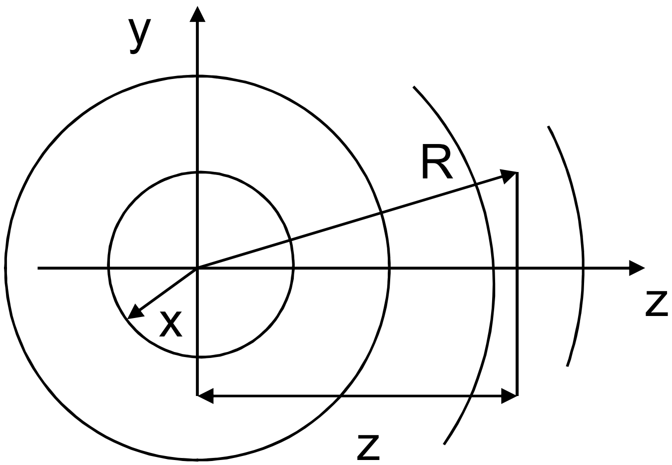
  $f < 0$ divergent



- So, if we know how to propagate through free space, then we can calculate field amplitude and phase through any imaging system

3.11 Huygens-Fresnel principle

- Spherical waves:
- Wavelet: $h = \dfrac{e^{ikR}}{R}$
- $R = \sqrt{x^2 + y^2 + z^2} = z\sqrt{1 + \dfrac{x^2 + y^2}{z^2}}$



- We are interested close to OA, i.e. small angles

$$\rightarrow \quad R \simeq z\left[1 + \frac{1}{2}\left(\frac{x^2 + y^2}{z^2}\right)\right] \qquad (3.39)$$

3.11 Huygens-Fresnel principle

- ! For amplitude $1/R \simeq 1/z$ is OK
- ! For phase $kR \simeq kz\left[1+\frac{1}{z}\left(\frac{x^2+y^2}{z^2}\right)\right]$

→ The wavelet becomes:

$$h(x,y) \simeq \frac{e^{ikz}}{z} e^{i\frac{k(x^2+y^2)}{2z}} \qquad (3.40a)$$

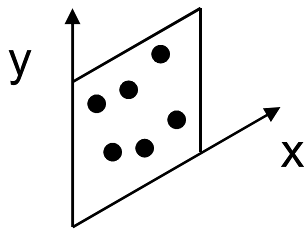
- Remember, for the lens we found:

$$t_e(x,y) = e^{i\phi_o} e^{i\frac{k}{2f}(x^2+y^2)} \qquad (3.40b)$$

- Free space acts on the wavefront like a divergent lens (note "+" sign in phase)

3.11 Huygens-Fresnel principle

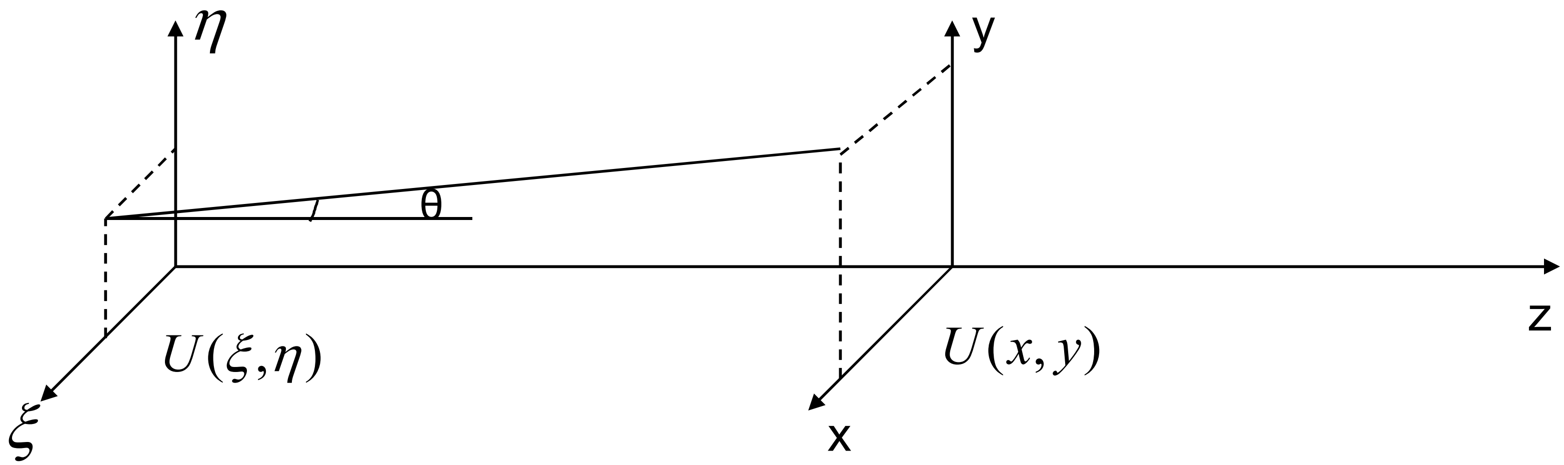
- At a given plane, a field is made of point sources



$$E(x,y)=\iint E(x',y')\delta(x-x')\delta(y-y')dx'dy'$$

- Eq 3.40 a-b represent the impulse response of the system (free space or lens)
- Recall linear systems (Chapter 2, page 12, Eq 2.16)
  - Final response (output) is the convolution of the input with the impulse response (or Greeen’s function)
- Nice! Space or time signals work the same!

3.11 Huygens-Fresnel principle



$$U(x,y) = \iint U(\xi,\eta) h(x-\xi, y-\eta)\, d\xi d\eta$$

$$= \iint U(\xi,\eta) e^{\frac{ik}{2z}\left[(x-\xi)^2 + (y-\eta)^2\right]} d\xi d\eta \qquad (3.41)$$

3.11 Huygens-Fresnel principle

- Fresnel diffraction equation = convolution
- Fresnel diffraction equation is an approximation $\left(R = z\left[1+\frac{x^2+y^2}{2z^2}\right]\right)$ of Huygens principle (17th century)

$$U(x,y) = \frac{1}{i\lambda}\int\int U(\xi,\eta)\frac{e^{ikR(\xi,\eta)}}{R(\xi,\eta)}\cos\theta(\xi,\eta)d\xi d\eta \quad (3.42)$$

- ! Fresnel is good enough for our pourpose
- Note: we don't care about constants A (no x-y dependence)

$$= \int\int U(\xi,\eta)e^{\frac{ik}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]}d\xi d\eta$$

3.12 Fraunhofer Approximation

- One more approximation (far field)
- The phase factor in Fresnel is:

$$\phi(x,y) = \frac{k}{2z}\left[(x-\xi)^2 + (y-\eta)^2\right] =$$

$$= \frac{k}{2z}\left[(x^2+y^2) + \underbrace{(\xi^2+\eta^2)}_{\approx 0} - 2(x\xi + y\eta)\right] \qquad (3.43)$$

- If $z >> k(\xi^2+\eta^2)$, we obtain the Fraunhofer equation:

$$U(x,y) = A\iint_{-\infty}^{\infty} U(\xi,\eta) e^{-\frac{2\pi}{\lambda z}(x\xi + y\eta)} d\xi d\eta \qquad (3.44)$$

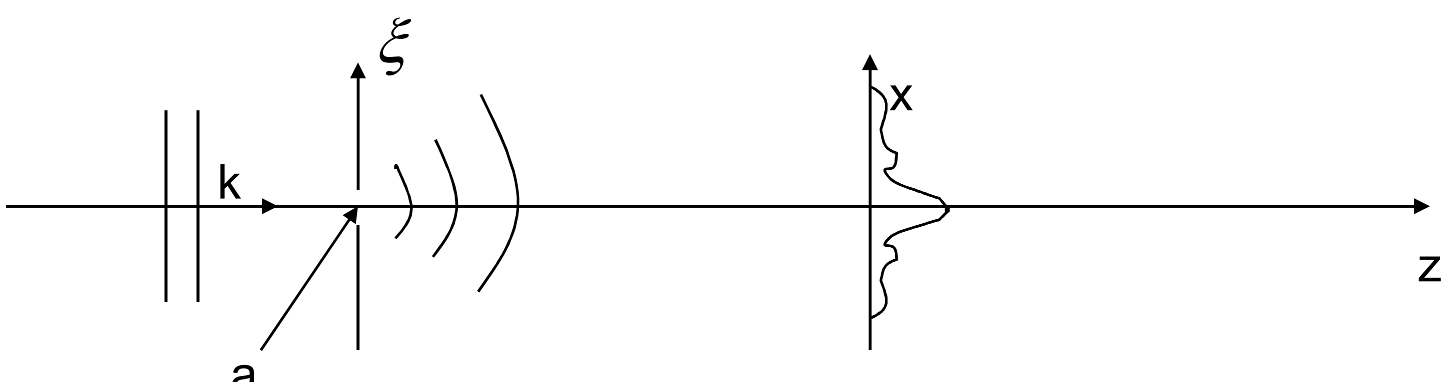
- Thus eq. 3.44 defines a Fourier transform
- Useful to calculate diffraction patterns !

3.12 Fraunhofer Approximation

- Let's define: $\begin{cases} f_x = \dfrac{x}{\lambda z} \\ f_y = \dfrac{y}{\lambda z} \end{cases}$

$$\rightarrow U(f_x, f_y) = \iint_{-\infty}^{\infty} U(\xi, \eta).e^{-i2\pi(\xi f_x + \eta f_y)} d\xi d\eta \qquad (3.45)$$

- Example: diffraction on a slit



3.12 Fraunhofer Approximation

- One dimensional: $U(x) = \Pi\left(\frac{x}{a}\right) =$ $\begin{cases} a, & |x| < a/2 \\ 0, & \text{rest} \end{cases}$

- The far-field is given by Fraunhofer eq:

$$U(f_x) = \int_{-\infty}^{\infty} U(x) e^{-i2\pi x f_x} dx =$$

$$= \Im\left[\Pi\left(\frac{x}{a}\right)\right] =$$

- Similarity Theorem + $\Im[\Pi(x)] = \sin c(f_x)$:

→ $U(f_x) = a \sin c(af_x) =$

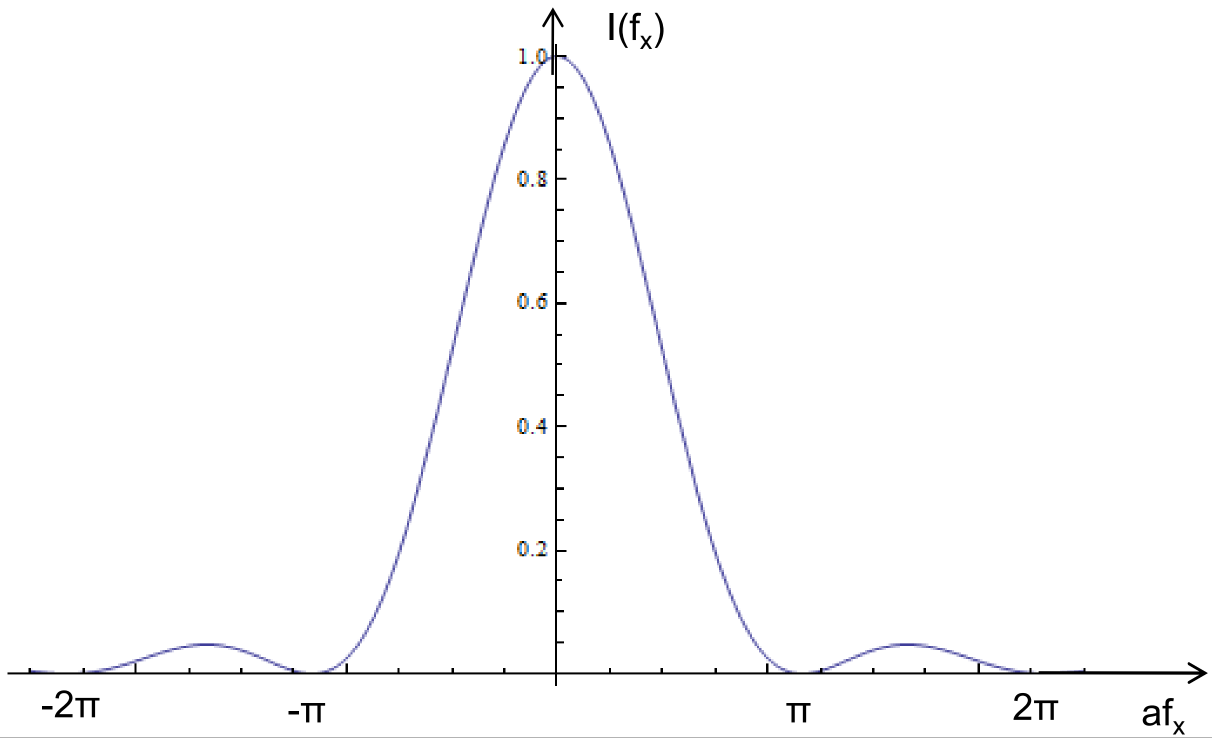
$$= a \frac{\sin(af_x)}{af_x} \qquad (3.46)$$

- $f_x = \frac{x}{\lambda z}$

3.12 Fraunhofer Approximation

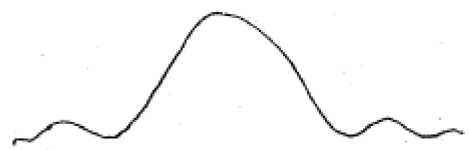
- Always measure intensity → the diffraction pattern is:

$$I(f_x) = |U(f_x)|^2 = a^2\left[\frac{\sin(af_x)}{af_x}\right]^2 \quad (3.48)$$

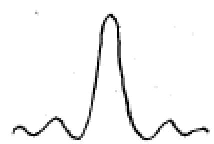


3.12 Fraunhofer Approximation

- Note: $\sin(af_x) = \sin\left(\frac{f_x}{\frac{1}{a}}\right)$ → $\frac{1}{a}$ = width of diffraction pattern
- narrow slit: a_1 →

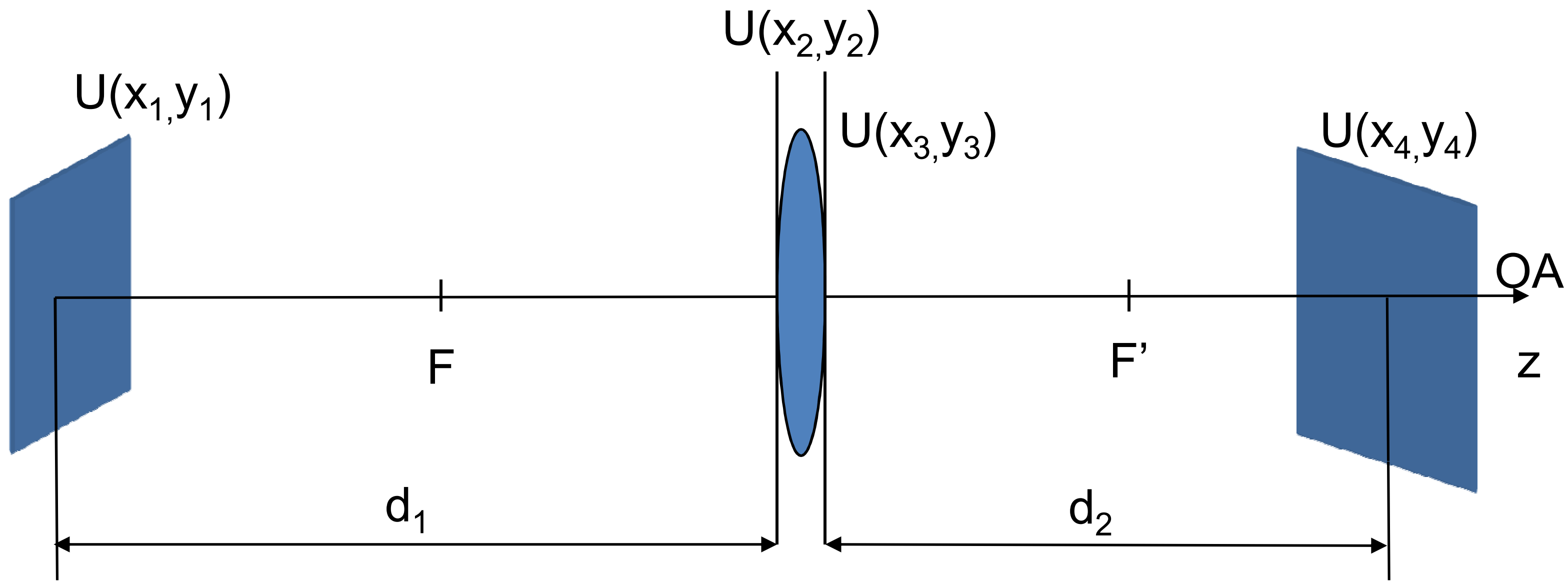


- wide slit: a_2 →



- Similarity Theorem ↔ uncertainty principle

3.13 Fourier Properties of lenses



- Propagation:

$$U(x_1,y_1) \xrightarrow{\text{Fresnel}} U(x_2,y_2)$$

$$U(x_2,y_2) \xrightarrow{\text{Lens}} U(x_3,y_3)$$

$$U(x_3,y_3) \xrightarrow{\text{Fresnel}} U(x_4,y_4)$$

3.13 Fourier Properties of lenses

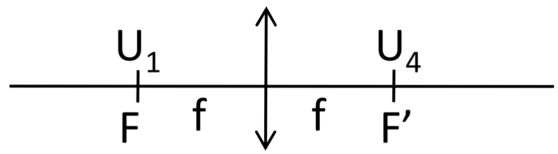
- $U(x_2,y_2)=A_{12}\iint U(x_1,y_1)e^{\frac{ik}{2d_1}\left[(x_2-x_1)^2+(y_2-y_1)^2\right]}dx_1dy_1$
- $U(x_3,y_3)=A_{23}U(x_3,y_3)e^{-i\frac{k}{2f}\left[x_3^2+y_3^2\right]}$
- $U(x_4,y_4)=A_{34}\iint U(x_3,y_3)e^{\frac{ik}{2d_2}\left[(x_4-x_3)^2+(y_4-y_3)^2\right]}dx_3dy_3$

(3.49)

- Combining Eqs (3.49) is a little messy, but there is a special case when eqs simplify → very useful

3.13 Fourier Properties of lenses

- If $d_1 = d_2 = f$

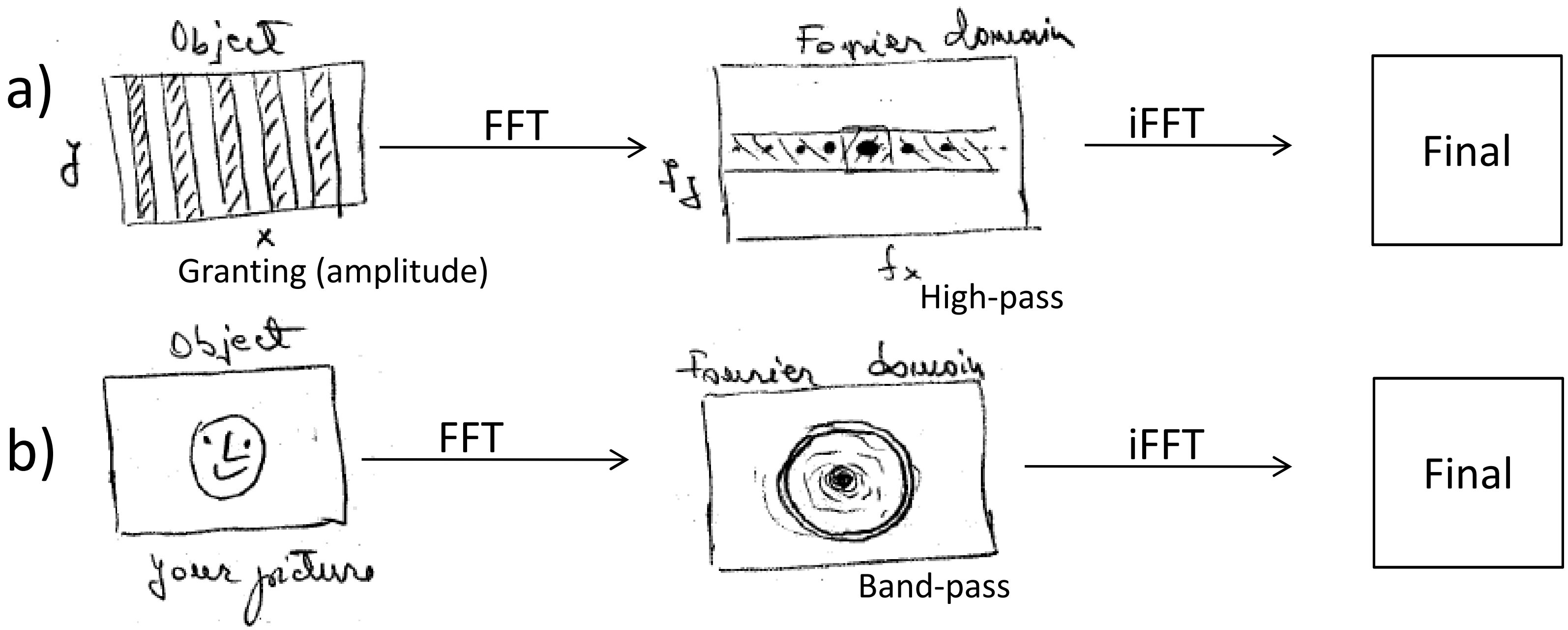


$$\begin{cases} U(x_4, y_4) = A_{41} \int \int_{-\infty}^{\infty} U_1(x_1, y_1) e^{-i2\pi(x_1 f_x + y_1 f_y)} dx_1 dy_1 \\ f_x = \frac{x_4}{\lambda f}; f_y = \frac{y_4}{\lambda f} \end{cases} \qquad (3.50)$$

- Same eq as (3.45); now <u>z → f</u>
- Lenses work as Fourier transformers
  - Useful for <u>spatial filtering</u>

3.13 Fourier Properties of lenses

- Exercise: Use Matlab to FFT images (look “fft2” in help)



- Note the relationship between the frequencies passed and the details / contrast in the final image